A non-local flow for Riemann surfaces

Matthew J. Gursky

Abstract  A non-local flow is defined for compact Riemann surfaces. Assuming the initial metric has positive Gauss curvature and is not conformal to the round sphere, the flow exists on some maximal time interval, and converges along a subsequence to a metric which admits a conformal Killing vector field. By a result of Tashiro (Trans Am Math Soc 117:251–275, 1965), the limiting metric must be conformal to the round sphere.

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1 Introduction

In this article we define a nonlocal geometric flow defined on compact Riemann surfaces. The metric evolution consists of two terms, the (normalized) Ricci flow and a nonlocal term. The purpose of introducing a non-local term is two-fold. First, it permits us to impose a nice evolution equation on the curvature. Second, the non-local term is defined by an equation whose solvability can be geometrically characterized. In particular, the non-local term is well defined except when the metric is pointwise conformal to the round sphere.

To make matters more precise, let us begin with Hamilton’s Ricci flow on surfaces [7]:

\[
\begin{align*}
\frac{\partial}{\partial t} g_{ij} &= (r - R) g_{ij}, \\
g(\cdot, 0)_{ij} &= (g_0)_{ij},
\end{align*}
\]

(1.1)

where

\[
r = \frac{\int R \, dV}{\int dV}
\]

(1.2)
is the mean value of the scalar curvature. To compute the evolution of the curvature under this flow, one applies the standard formula for the linearized scalar curvature operator $dR$ (see [2, Chap. 3]):

$$dR(g)[h] = \frac{d}{ds} R(g + sh) \bigg|_{s=0} = -\Delta (\text{tr}h) + \delta^2 h - \langle \text{Ric}, h \rangle.$$  

(1.3)

For the Ricci flow, $h = -(r - R)g$; thus the scalar curvature evolves by

$$\frac{\partial}{\partial t} R = \Delta R + R(R - r).$$  

(1.4)

At the risk of oversimplifying, one might say that there are two main difficulties associated to the study of the Ricci flow on surfaces with positive curvature. The first is the presence of the quadratic nonlinearity (the ‘reaction’ term) in (1.4). Hamilton used a parabolic Harnack inequality along with an entropy estimate to show that the curvature remains bounded, a fact which is not at all obvious. The second difficulty is the presence of solitons: only recently did a proof appear which classified solitons without using the uniformization theorem (see [3]). Of course, neither of these issues arise in the case of non-positive Euler characteristic; indeed, the quadratic term is actually helpful in the case of negative curvature. Therefore, in this paper we only consider the case of positive Euler characteristic. To overcome the first difficulty we will introduce a nonlocal term into the Ricci flow, in order to impose a nicer evolution equation for the curvature. The form of this non-local term is actually inspired by the second difficulty, as we now explain.

Let $\mathcal{H}_g : C^\infty(M^2) \to S^2 T^*M^2$ denote the trace-free Hessian operator

$$\mathcal{H}_g u = \nabla^2 u - \frac{1}{2} (\Delta u) g.$$  

(1.5)

Consider the evolution equation

$$\frac{\partial}{\partial t} g = (r - R)g + \mathcal{H} u,$$  

(1.6)

where $u$ is to be determined. The evolution of the scalar curvature under this flow is

$$\frac{\partial}{\partial t} R = dR(g)[(r - R)g + \mathcal{H} u]$$

$$= \Delta R + R(R - r) + dR(g)[\mathcal{H} u]$$

$$= \Delta R + R(R - r) + A u,$$  

(1.7)

where the operator $A = A_g$ is defined by

$$Au = dR(g)[\mathcal{H} u]$$

$$= \frac{1}{2} \Delta^2 u + \frac{1}{2} \nabla^i (R \nabla_i u).$$  

(1.8)

The idea is to choose $u$ in order to cancel the nonlinearity in (1.7). To this end, we couple the flow (1.6) to the following defining equation for $u$:

$$Au = -r(R - r) - R^2 + r_2,$$  

(1.9)

where

$$r_2 = \frac{\int R^2 \, dV}{\int dV}.$$  

(1.10)