Improving Pogorelov’s isometric embedding counterexample

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Abstract We construct a $C^{2,1}$ metric of non-negative Gauss curvature with no $C^2$ local isometric embedding in $\mathbb{R}^3$.

1 Introduction

In this note, we modify Pogorelov’s $C^{2,1}$ Riemannian metric in [8] with no local $C^2$ isometric embedding in $\mathbb{R}^3$ and sign changing Gauss curvature to a similar one with non-negative Gauss curvature. Though each effective piece of Pogorelov’s metric upon which he drew his contradiction has no negative Gauss curvature, inevitably the Gauss curvature must be negative somewhere, as each effective piece of the metric is surrounded by a flat one with Euclidean metric. We add “tails” to those effective pieces and construct matching metric on the tails. Then we obtain the final metric $g$ with $K_g \geq 0$ admitting no local $C^2$ isometric embedding. To be precise, we state

Theorem 1.1 There exists a $C^{2,1}$ metric $g$ in $B_1 \subset \mathbb{R}^2$ with Gauss curvature $K_g \geq 0$ such that there is no $C^2$ isometric embedding of $(B_r(0), g)$ in $\mathbb{R}^3$ for any $r > 0$.

Remark As Jacobowitz observed [4, p.249], one can improve Pogorelov’s $C^{2,1}$ metric to $C^{3,\alpha}$ with no $C^2,\beta$ local isometric embedding for $1 > \beta > 2\alpha$. We can also modify this $C^{3,\alpha}$
metric so that its Gauss curvature is non-negative and it still admits no \( C^{2,\beta} \) local isometric embedding in \( \mathbb{R}^3 \). Instead of (2.1), we take

\[
 u = \int_a^r \left( s - \frac{a}{2} \right)^{2+\alpha} \frac{(2+\alpha) s}{2} ds \quad \frac{a}{2} \leq r < a
\]

and repeat Steps 1 to 4 in Sect. 2. Assuming the existence of \( C^{2,\beta} \) isometric embedding, the contradiction inequality (2.4) becomes

\[
 h_{22} (t, b) \geq \frac{\eta (k)}{100 \cdot 2^\beta a^\alpha \| h \|_{C^{2,\beta}}} \frac{1}{c^{\beta-2\alpha}} \rightarrow \infty \quad \text{as } c \rightarrow 0.
\]

On the other hand, Lin [5] showed that there exists a \( C^{k-8} \) isometric embedding of \(( B_{rk}, g ) \) in \( \mathbb{R}^3 \) for the \( C^k \) metric \( g \) with \( k \geq 12 \) and \( K_g \geq 0 \); see also [2, Theorem 6.01]. For any sufficiently smooth metric with Gauss curvature changing sign cleanly, a sufficiently smooth local isometric embedding in \( \mathbb{R}^3 \) was obtained in Lin [6]; see also a simplified proof by Han [1]. For any smooth metric with Gauss curvature changing sign cleanly, a smooth local isometric embedding in \( \mathbb{R}^3 \) was derived by Nakamura and Maeda [7]. For smooth metrics of non-positive Gauss curvatures with certain non-degeneracy, Han, Hong, and Lin [3] derived a local smooth isometric embedding in \( \mathbb{R}^3 \).

Now one further problem is the existence of local smooth isometric embedding of any smooth Riemannian metric with non-negative curvature. More importantly, still remains the basic question of the existence in \( \mathbb{R}^3 \) of local \( C^\infty \), or even \( C^2 \) isometric embedding of arbitrary \( C^\infty \) Riemannian metrics. For a systematic study and references of the isometric embedding, we refer to the recent book by Han and Hong [2].

2 Construction

**Step 1.** We first define a metric \( g = e^{2u} dx^2 \) in \( B_a (0) \) on \( x = (x_1, x_2) \) plane for small positive \( a \), where

\[
 u = \left\{ \begin{array}{ll}
 0 & 0 \leq r < \frac{a}{2} \\
 \int_{\frac{a}{2}}^r \left( s - \frac{a}{2} \right)^2 ds & \frac{a}{2} \leq r < a
\end{array} \right.
\]

and \( r = |x| \). Then the metric is \( C^{2,1} \) inside \( B_a \) and smooth away from \( \partial B_{a/2} \). The Gauss curvature \( K_g = -e^{-2u} \triangle u = -e^{-2u} \frac{(ru)^2}{r} \) satisfies

\[
 K_g = \left\{ \begin{array}{ll}
 1 + O \left( a^{-2} \right) & \frac{a}{2} \leq r < a \\
 0 & 0 \leq r < \frac{a}{2}
\end{array} \right.
\]

(2.2)

**Step 2.** We next modify the metric \( g \) or rather \( u \) outside \( B_{3a}^3 \). Let \( \Omega_a \) be a domain satisfying

\[
 \left\{ \begin{array}{l}
 B_{\frac{3a}{2}} \subseteq \Omega_a \\
 \Omega_a \subseteq [-a, a] \times [-1, 2] \\
 \partial \Omega_a \cap \{ x_2 = 2 \} = \text{a segment with positive length}
\end{array} \right.
\]