Global solutions of the heat equation with a nonlinear boundary condition

Kazuhiro Ishige · Tatsuki Kawakami

Received: 13 July 2009 / Accepted: 13 January 2010 / Published online: 4 February 2010
© Springer-Verlag 2010

Abstract We consider the heat equation with a nonlinear boundary condition,

\[ \begin{cases}
\partial_t u = \Delta u, & x \in \Omega, \quad t > 0, \\
\partial_{\nu} u = u^p, & x \in \partial \Omega, \quad t > 0, \\
u(x, 0) = \phi(x), & x \in \Omega,
\end{cases} \]

where \( \Omega = \{ x = (x', x_N) \in \mathbb{R}^N : x_N > 0 \}, N \geq 2, \partial_t = \partial/\partial t, \partial_{\nu} = -\partial/\partial x_N, p > 1 + 1/N, \) and \((N - 2)p < N\). In this paper we give a complete classification of the large time behaviors of the nonnegative global solutions of \((P)\).

Mathematics Subject Classification (2000) 35B40 · 35K55 · 35K60

1 Introduction

We consider the heat equation in the half space of \( \mathbb{R}^N \) with a nonlinear boundary condition,

\[ \begin{cases}
\partial_t u = \Delta u, & x \in \Omega, \quad t > 0, \\
\partial_{\nu} u = u^p, & x \in \partial \Omega, \quad t > 0, \\
u(x, 0) = \phi(x), & x \in \Omega,
\end{cases} \]

where \( \Omega = \{ x = (x', x_N) \in \mathbb{R}^N : x_N > 0 \}, N \geq 2, \partial_t = \partial/\partial t, \partial_{\nu} = -\partial/\partial x_N, \) and \( p > 1 \).

In this paper we assume that

\( \phi \in X \equiv \left\{ f \in L^\infty(\Omega) \cap L^2 \left( \Omega, e^{\frac{|x|^2}{4}}dx \right) : f \geq 0 \text{ in } \Omega \right\}, \quad 1 + 1/N < p, \quad (N - 2)p < N, \)

Communicated by Y. Giga.
and give a classification of the large time behaviors of the nonnegative global solutions of (1.1)–(1.3).

The nonlinear boundary value problem (1.1)–(1.3) can be physically interpreted as a non-linear radiation law, and has been studied in many papers (see [2–4,6–9,11,12,17,20–22], and references therein). However, for the large time behaviors of the solutions of (1.1)–(1.3) in unbounded domains, there are only a few papers even if $\Omega = \mathbb{R}_+^N$. Among others, in [4], Deng, Filo, and Levine proved that, if $1 < p \leq 1 + 1/N$, then there does not exist non-trivial global solutions of (1.1)–(1.3). Furthermore they proved that, if $p > 1 + 1/N$, then, for some “small” initial data $\phi$, there exists a non-trivial global solution of (1.1)–(1.3) satisfying

$$||u(t)||_{L^\infty(\Omega)} = O(t^{-1/2(p-1)}) \quad \text{as} \quad t \to \infty.$$  

Recently, in [17], the second author of this paper proved that there exists a positive constant $\delta$ with the following property:

$$\text{if} \|\phi\|_{L^1(\Omega)} \|\phi\|_{L^\infty(\Omega)}^{N(p-1)/2} < \delta, \text{ then there exists a global solution} \ u \ \text{of (1.1)–(1.3)}$$

such that $$||u(t)||_{L^q(\Omega)} = O(t^{-(N/2)(1-1/q)}) \quad \text{as} \quad t \to \infty \quad \text{for any} \ q \in [1, \infty].$$  

Furthermore he proved that there exists the limit

$$c_\ast = 2 \lim_{t \to \infty} \int_{\Omega} u(x,t)dx = 2 \left( \int \int_{\Omega} u(x,0)dx + \int_0^\infty \int_\Omega u(x,t)p \sigma dt \right) \quad \text{(1.7)}$$

such that

$$\lim_{t \to \infty} t^\frac{N}{2}(1-\frac{1}{q}) ||u(t) - c_\ast g(t)||_{L^q(\Omega)} = 0$$

for any $q \in [1, \infty]$, where $g(x,t) = (4\pi t)^{-N/2} \exp(-|x|^2/4t)$ (see also Proposition 2.1).

On the other hand, for the Cauchy problem of the semilinear heat equation,

$$\partial_t u = \Delta u + u^p \quad \text{in} \quad \mathbb{R}_+^N \times (0, \infty), \quad u(x,0) = \lambda \varphi \quad \text{in} \quad \mathbb{R}_+^N, \quad (1.8)$$

in [18], Kawanago gave a classification of the large time behaviors of the global solutions. He proved that, if $p > 1 + 2/N$ and $(N - 2)p < N + 2$, for any $\varphi \in X\setminus\{0\}$, there exists a positive constant $\lambda_\varphi$ such that

(a) if $0 < \lambda < \lambda_\varphi$, then the solution $u$ of (1.8) exists globally in time and $||u(t)||_{L^\infty(\mathbb{R}_+^N)} \asymp t^{-\frac{N}{2}}$ as $t \to \infty$;

(b) if $\lambda = \lambda_\varphi$, then the solution $u$ of (1.8) exists globally in time and $||u(t)||_{L^\infty(\mathbb{R}_+^N)} \asymp t^{-\frac{1}{p-1}}$ as $t \to \infty$;

(c) if $\lambda > \lambda_\varphi$, then the solution $u$ of (1.8) does not exist globally in time, and blows-up in a finite time, that is, $\limsup_{t \to T_M - 0} ||u(t)||_{L^\infty(\mathbb{R}_+^N)} = \infty$ for some $T_M > 0$.

(see also [16]). Furthermore he proved that there exists a positive constant $\delta' > 0$ with the following property:

$$\text{if} \|\phi\|_{L^N(\mathbb{R}_+^N)}^{N(p-1)/2} < \delta', \text{ then there exists a global solution} \ u \ \text{of (1.8)}$$

such that $||u(t)||_{L^q(\Omega)} = O(t^{-(N/2)(1-1/q)}) \quad \text{as} \quad t \to \infty \quad \text{for any} \ q \in [1, \infty].$  

The property (1.9) plays an important role of proving the existence of $\lambda_\varphi$.

In this paper, by following the strategy in [16] and [18], we study the nonlinear boundary problem (1.1)–(1.3) under the conditions (1.4) and (1.5), and give a classification of the large time behaviors of the nonnegative global solutions. Furthermore we