A Hardy type inequality for $W^{m,1}(0,1)$ functions

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Abstract In this paper, we consider functions $u \in W^{m,1}(0,1)$ where $m \geq 2$ and $u(0) = Du(0) = \cdots = D^{m-1}u(0) = 0$. Although it is not true in general that $\frac{D^j u(x)}{x^{m-j}} \in L^1(0,1)$ for $j \in \{0, 1, \ldots, m-1\}$, we prove that $\frac{D^j u(x)}{x^{m-j-k}} \in W^{k,1}(0,1)$ if $k \geq 1$ and $1 \leq j + k \leq m$, with $j, k$ integers. Furthermore, we have the following Hardy type inequality,

$$\left\| D^k \left( \frac{D^j u(x)}{x^{m-j-k}} \right) \right\|_{L^1(0,1)} \leq \frac{(k-1)!}{(m-j-1)!} \left\| D^m u \right\|_{L^1(0,1)},$$

where the constant is optimal.

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1 Introduction

It is well known [1] that if $u \in W^{1,p}(0,1)$ and $u(0) = 0$ then the so called Hardy inequality holds for $p > 1$, that is

$$\int_0^1 \left| \frac{u(x)}{x} \right|^p dx \leq \left( \frac{p}{p-1} \right)^p \int_0^1 \left| u'(x) \right|^p dx.$$

(1)

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The constant $\frac{p}{p+1}$ is optimal for this inequality and it blows up as $p$ goes to 1. This behaviour is confirmed by the fact that no such inequality can be proved when $p = 1$, as we can consider (see e.g. [2]) the non-negative function on $(0,1)$ defined by

$$u(x) = \frac{1}{1 - \log x}.$$  

A simple computation shows that this function belongs to $W^{1,1}(0,1)$, $u(0) = 0$, but $\frac{u(x)}{x}$ is not integrable.

When we turn to functions $u \in W^{2,p}(0,1)$, $p \geq 1$, with $u(0) = u'(0) = 0$, there are three natural quantities to consider: $\frac{u(x)}{x}$, $\frac{u'(x)}{x}$ and $(\frac{u(x)}{x})' = \frac{u'(x)}{x} - \frac{u(x)}{x^2}$. If $p > 1$, it is clear that both $\frac{u'(x)}{x}$ and $\frac{u(x)}{x^2} = \frac{u'(x)}{x} - \frac{1}{x^2} \int_0^x t u''(t) \, dt$ belong to $L^p(0,1)$. Thus $(\frac{u(x)}{x})' \in L^p(0,1)$.

If $p = 1$ one can no longer assert that $\frac{u(x)}{x}$, $\frac{u'(x)}{x}$ belong to $L^1(0,1)$, but surprisingly $(\frac{u(x)}{x})' \in L^1(0,1)$. This reflects a “magic” cancellation of the non-integrable terms in the difference $$(\frac{u(x)}{x})' = \frac{u'(x)}{x} - \frac{u(x)}{x^2}.$$ The same phenomenon remains valid when we keep increasing the number of derivatives, and this is the main result of this paper.

**Definition 1.1** We say that $u$ has the property $(P_m)$ if

$$u \in W^{m,1}(0,1) \text{ and } u(0) = Du(0) = \cdots = D^{m-1}u(0) = 0,$$

where $D^i u$ denotes the $i$-th derivative of $u$.

**Theorem 1.2** If $u$ has the property $(P_m)$ and $j$, $k$ are non-negative integers, then

1. If $k \geq 1$ and $1 \leq j + k \leq m$ then $\frac{D^j u(x)}{x^{m-j-k}}$ has the property $(P_k)$ and

$$\left\| D^k \left( \frac{D^j u(x)}{x^{m-j-k}} \right) \right\|_{L^1(0,1)} \leq \frac{(k-1)!}{(m-j-1)!} \| D^m u \|_{L^1(0,1)}. \tag{3}$$

The constant being the best possible.

2. There exists $w$ having the property $(P_m)$ such that

$$\frac{D^j w(x)}{x^{m-j}} \notin L^1(0,1) \text{ for all } j \in \{0, 1, \ldots, m-1\}. \tag{4}$$

**Remark 1.1** For functions $u \in W^{2,p}(0,1)$, $p > 1$, with $u(0) = u'(0) = 0$, a slightly stronger result holds, namely, when we estimate the $L^p$ norms of the three quantities $\frac{u(x)}{x^2}$, $\frac{u'(x)}{x}$ and $(\frac{u(x)}{x})'$, we obtain

$$\left\| \frac{u(x)}{x^2} \right\|_p \leq \alpha_p \left\| u'' \right\|_p, \quad \left\| \frac{u'(x)}{x} \right\|_p \leq \beta_p \left\| u'' \right\|_p, \quad \text{and} \quad \left\| \left( \frac{u(x)}{x} \right)''' \right\|_p \leq \gamma_p \left\| u'' \right\|_p, \tag{5}$$

with $\alpha_p$, $\beta_p$, $\gamma_p$ be the best possible constants. It is easy to see that $\alpha_p \to \infty$, $\beta_p \to \infty$ when $p$ approaches 1. However, a similar “magic” cancellation appears and $\gamma_p$ remains bounded as $p$ goes to 1. A proof of this latter fact is presented in Sect. 3.