On the stability of Mañé critical hypersurfaces

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Abstract We construct examples of Tonelli Hamiltonians on $\mathbb{T}^n$ (for any $n \geq 2$) such that the hypersurfaces corresponding to the Mañé critical value are stable (i.e. geodesible). We also provide a criterion for instability in terms of closed orbits in free homotopy classes and we show that any stable energy level of a Tonelli Hamiltonian must contain a closed orbit.

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1 Introduction

Let $M$ be a closed manifold and $H : T^*M \to \mathbb{R}$ a Tonelli Hamiltonian, that is, a smooth function that is strictly fiberwise convex and superlinear. The latter means that for all $x \in M$ and some (and hence any) Riemannian metric

$$\lim_{|p| \to \infty} \frac{H(x, p)}{|p|} = \infty.$$ 

Given a covering $\Pi : \hat{M} \to M$ we can associate to $H$ a Mañé critical value as follows. We consider the lift $\hat{H}$ of $H$ to $\hat{M}$ and we set

$$c(\hat{H}) := \inf_{u \in C^\infty(\hat{M}, \mathbb{R})} \sup_{x \in \hat{M}} \hat{H}(x, d_x u).$$

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If \( \tilde{M} \) is a covering of \( \hat{M} \), then clearly \( c(H) \leq c(\tilde{H}) \) and equality holds if \( \tilde{M} \) is a finite cover of \( M \). If \( \hat{M} \) is the universal covering of \( M \), we will denote by \( c_u(H) \) the corresponding critical value. If we consider the abelian cover given by the kernel of the Hurewicz map \( \pi_1(M) \leftrightarrow H_1(M, \mathbb{R}) \) we obtain what is called the strict critical value. We will denote it by \( c_0(H) \) and it coincides with \( -\beta(0) \), where \( \beta : H_1(M, \mathbb{R}) \to \mathbb{R} \) is Mather’s minimal action function in homology [7]. Clearly \( c_u(H) \leq c_0(H) \) but in general the inequality can be strict (this fact was first pointed out in [9], but see [1] for many more examples). On the other hand, \( c_u(H) = c_0(H) \) as soon as \( \pi_1(M) \) is amenable [5].

Let \( e \) denote the smallest value of \( k \) such that \( \Sigma_k := H^{-1}(k) \) intersects every fibre of \( T^*M \). We will suppose throughout that the critical values \( c_u(H) \) and \( c_0(H) \) are strictly bigger than \( e \); this ensures in particular that they are regular values of \( H \). This happens rather frequently, e.g. if \( H \) has the form \( H(x, p) = \frac{1}{2}|p + \theta_x|^2 \), where \( \theta \) is a 1-form which is not closed.

The critical values have significance from the point of view of the symplectic topology of the hypersurfaces \( \Sigma_k [1,10] \). For example, it is well known that if \( k > c_0(H) \), then \( \Sigma_k \) is of contact type. It is also known that if \( M \neq \mathbb{T}^2 \), then \( \Sigma_k \) is not of contact type for \( k \in [c_u(H), c_0(H)] \) (cf. [2, Theorem B.1]) and it is an open problem to show that in fact \( \Sigma_k \) is never of contact type for \( k \in (e, c_u(H)) \). Evidence in favour of a positive answer to this problem when \( \dim M = 2 \) is given in [4]. For the case of \( \mathbb{T}^2 \) the situation is a bit exceptional due to the fact that this is the only case where the projection map \( H_1(\Sigma_k, \mathbb{R}) \leftrightarrow H_1(M, \mathbb{R}) \) is not injective. Exploiting this fact, an example is given in [4] with the property that the Mañé critical hypersurface \( \Sigma_c \), where \( c = c_u(H) = c_0(H) \), is of contact type. On the other hand \( \Sigma_c \) can never be of restricted contact type.

This paper is prompted by the recent interest in a weaker condition than contact type, namely, the stability or geodesibility of these hypersurfaces [1]. Recall that \( \Sigma_k \) is said to be stable if there exists a smooth 1-form \( \lambda \) such that vectors \( v \neq 0 \) tangent to the characteristic foliation of \( \Sigma_k \) annihilate \( d\lambda \) but \( \lambda(v) \neq 0 \). This is equivalent to saying that the characteristic foliation of \( \Sigma_k \) is geodesible, i.e., there exists a smooth Riemannian metric on \( \Sigma_k \) such that the leaves of the foliation are geodesics of the metric. The terminology “stable” was coined by Hofer and Zehnder [6] who introduced the notion via another equivalent definition: the hypersurface \( \Sigma_k \) is stable if a neighborhood of it can be foliated by hypersurfaces whose characteristic foliations are conjugate. We remark that in general these nearby hypersurfaces do not need to coincide with \( H^{-1}(r) \) for \( r \) near \( k \).

The question that arises now is: if \( M \neq \mathbb{T}^2 \), can the Mañé critical hypersurfaces \( \Sigma_{c_u} \) and \( \Sigma_{c_0} \) ever be stable? To motivate our result, let us consider an illustrative class of Hamiltonians first. Suppose \( H(x, p) = \frac{1}{2}|p + \theta_x|^2 \), where the 1-form \( \theta \) has the property that \( |\theta_x| = 1 \) for all \( x \in M \). Moreover, suppose that the vector field \( X \) on \( M \) metric-dual to \( \theta \) has an invariant probability measure \( \mu \) with zero homology, that is,

\[
\int_M \omega(X) \, d\mu = 0
\]

for any closed 1-form \( \omega \). This is equivalent to saying that the flow of \( X \) has no cross-section [11]. Since the zero section \( p = 0 \) sits inside \( \Sigma_{1/2} \), we see that \( c_0(H) \leq 1/2 \). The characteristic foliation of \( \Sigma_{1/2} \) is tangent to the zero section and restricted to it the dynamics coincides with that of \( X \). The condition that \( X \) has an invariant probability measure \( \mu \) with zero homology forces \( c_0(H) = 1/2 \). To see this, let \( \xi \) be the Liouville 1-form of \( T^*M \). Since \( \xi \) vanishes on the zero section and the characteristic foliation has an invariant measure with zero homology supported on \( p = 0 \), \( \Sigma_{1/2} \) cannot be of contact type. But we know that \( \Sigma_k \) is of contact type for any \( k > c_0(H) \), thus \( c_0(H) = 1/2 \). We already pointed out that the dynamics of \( X \) sits inside the dynamics of the characteristic foliation, thus if \( X \) is not