Balanced split sets and Hamilton–Jacobi equations

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Abstract We study the singular set of solutions to Hamilton–Jacobi equations with a Hamiltonian independent of $u$. In a previous paper, we proved that the singular set is what we called a balanced split locus. In this paper, we find and classify all balanced split loci, identifying the cases where the only balanced split locus is the singular locus, and the cases where this does not hold. This clarifies the relationship between viscosity solutions and the classical approach of characteristics, providing equations for the singular set. Along the way, we prove more structure results about the singular sets.

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1 Introduction

In this paper we consider the following boundary value problem:

\begin{align}
H(p, du(p)) &= 1 \quad p \in \Omega \\
u(p) &= g(p) \quad p \in \partial \Omega
\end{align}

for a smooth compact manifold $\Omega$ of dimension $n$ with boundary, $H$ smooth, $H^{-1}(1) \cap T_p^* \Omega$ strictly convex for every $p$, and $g$ smooth and satisfying the compatibility condition:

\[ |g(y) - g(z)| < d(y, z) \quad \forall y, z \in \partial \Omega \]

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where $d$ is the distance induced by the Finsler metric:

$$
\varphi_p(v) = \sup \left\{ \langle v, \alpha \rangle_p : \alpha \in T^*_p \Omega, H(p, \alpha) = 1 \right\}
$$

This definition gives a norm in every tangent space $T_p \Omega$. Indeed, $H$ is a norm at every tangent space if we make the harmless assumption that $H$ is positively homogeneous of order 1: $H(p, \lambda \alpha) = \lambda H(p, \alpha)$ for $\lambda > 0$.

A unique viscosity solution is given by the Lax–Oleinik formula:

$$
u(p) = \inf_{q \in \partial \Omega} \{ d(p, q) + g(q) \} \tag{1.5}$$

A local classical solution can be computed near $\partial \Omega$ following characteristic curves, which are geodesics of the metric $\varphi$ starting from a point in $\partial \Omega$ with initial speed given by a vector field on $\partial \Omega$ that is determined by $H$ and $g$ (see 3.1): if $\gamma : [0, t) \to \Omega$ is the unique (projected) characteristic from a point $q \in \partial \Omega$ to $p = \gamma(t)$ that does not intersect $\text{Sing}$, then $u(p) = g(q) + t$. The viscosity solution can be thought of as a way to extend the classical solution to the whole $\Omega$.

Let $\text{Sing}$ be the closure of the singular set of the viscosity solution $u$ to the above problem. $\text{Sing}$ has a key property: any point in $\Omega \setminus \text{Sing}$ can be joined to $\partial \Omega$ by a unique characteristic curve that does not intersect $\text{Sing}$. A set with this property is said to split $\Omega$. Once characteristic curves are known, if we replace $\text{Sing}$ by any set $S$ that splits $\Omega$, we can still apply the formula in the last paragraph to obtain another function with some resemblance to the viscosity solution (see Definition 2.4).

The set $\text{Sing}$ has an extra property: it is a balanced split locus. This notion, introduced in [1] and inspired originally by the paper [10], is related to the notion of semiconcave functions that is now common in the study of Hamilton–Jacobi equations (see Sect. 2.1). Our goal in this paper is to determine whether there is a unique balanced split locus. In the cases when this is not true, we also give an interpretation of the multiple balanced split loci.

Finally, we recall that the distance function to the boundary in Riemannian and Finsler geometry is the viscosity solution of a Hamilton–Jacobi equation [13], and the cut locus is the closure of the singular set of the distance function to the boundary [11]. Thus, our results also apply to cut loci in Finsler geometry.

1.1 Outline

In Sect. 2 we state our results, give examples, and comment on possible extensions. Section 3 gathers some of the results from the literature we will need, and includes a few new lemmas that we use later. Section 4 contains our proof that the distance to a balanced split locus and distance to the k-th conjugate point are Lipschitz. Section 5 contains the proof of the main theorems, modulo a result that is proved in Sect. 6. This last section also features detailed descriptions of a balanced split set at each of the points in the classification introduced in [1].

2 Statement of results

2.1 Setting

We study a Hamilton–Jacobi equation given by (1.1) and (1.2) in a $C^\infty$ compact manifold with boundary $\Omega$. $H$ is smooth and strictly convex in the second argument and the boundary data $g$ is smooth and satisfies (1.3).