Existence of solutions for quasilinear elliptic equations with jumping nonlinearities under the Neumann boundary condition

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Abstract By variational methods, we prove the existence of a non-trivial solution for the quasilinear elliptic equations with jumping nonlinearities under the Neumann boundary condition. We also provide existence results for positive, negative and non-trivial multiple solutions. The studied equations contain the \( p \)-Laplacian problems as a special case.

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1 Introduction and statements of main results

1.1 Introduction

In this paper, we consider the existence of a non-trivial solution for the following quasilinear elliptic equation

\[
\begin{aligned}
-\text{div} \, A(x, \nabla u) &= f(x, u) \quad \text{in } \Omega, \\
\frac{\partial u}{\partial v} &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]  

(P)

where \( v \) denotes the outward unit normal vector on \( \partial \Omega \), \( 1 < p < \infty \), \( \Omega \subset \mathbb{R}^N \) is a bounded domain with \( C^2 \) boundary \( \partial \Omega \). Here, \( A : \bar{\Omega} \times \mathbb{R}^N \rightarrow \mathbb{R}^N \) is a map which is strictly monotone in the second variable and satisfies certain regularity conditions (see the following assumption \( (A) \)). The Eq. (P) contains the corresponding \( p \)-Laplacian problem as a special case.
However, in general, we do not suppose that this operator is \((p - 1)\)-homogeneous in the second variable. Here, we say that \(u \in W^{1,p}(\Omega)\) is a (weak) solution of (P) if
\[
\int_{\Omega} A(x, \nabla u) \nabla \varphi \, dx = \int_{\Omega} f(x, u) \varphi \, dx
\]
for all \(\varphi \in W^{1,p}(\Omega)\).

Throughout this paper, we assume that the operator \(A\) and the nonlinear term \(f\) satisfy the following assumption (A) and (F), respectively:
\[(A)\]
\[
A(x, y) = a(x, |y|)y, \quad \text{where } a(\cdot, t) > 0 \text{ for all } (x, t) \in \tilde{\Omega} \times (0, +\infty) \text{ and } A \in C_{\text{loc}}^{0,\gamma}(\tilde{\Omega} \times \mathbb{R}^N, \mathbb{R}^N) \cap C^{1}(\tilde{\Omega} \times (\mathbb{R}^N \setminus \{0\}), \mathbb{R}^N) \text{ with some } 0 < \gamma \leq 1;
\]
(i) there exist \(C_1 > 0\) and \(1 < p < \infty\) such that
\[
|D_y A(x, y)| \leq C_1 |y|^{p-2} \text{ for every } x \in \tilde{\Omega}, \text{ and } y \in \mathbb{R}^N \setminus \{0\};
\]
(ii) there exist \(C_2 > 0\) and \(1 < q < \infty\) such that
\[
|D_y A(x, y)x| \leq C_2 |y|^{q-2} |x|^2 \text{ for every } x \in \tilde{\Omega}, \text{ and } y \in \mathbb{R}^N \setminus \{0\} \text{ and } x \in \mathbb{R}^N.
\]
(iii) there exists a \(C_0 > 0\) such that
\[
D_y A(x, y)\xi \cdot \xi \geq C_0 |y|^{p-2} |\xi|^2 \text{ for every } x \in \tilde{\Omega}, \quad y \in \mathbb{R}^N \setminus \{0\} \text{ and } \xi \in \mathbb{R}^N.
\]

\[(F)\]
\(f\) is a Carathéodory function on \(\Omega \times \mathbb{R}\) with \(f(x, 0) = 0\) for a.e. \(x \in \Omega\) and satisfies the following conditions for some constants \(\alpha_0, \beta_0, \alpha\) and \(\beta\):
\[
f(x, u) = \begin{cases} 
\alpha_0 u_+^{p-1} - \beta_0 u_-^{p-1} + h_0(x, u), \\
\alpha u_+^{p-1} - \beta u_-^{p-1} + h(x, u), 
\end{cases}
\]
\[
h_0(x, u) = o(|u|^{p-1}) \quad \text{as } |u| \to 0, \quad \text{uniformly in a.e. } x \in \Omega,
\]
\[
h(x, u) = o(|u|^{p-1}) \quad \text{as } |u| \to \infty, \quad \text{uniformly in a.e. } x \in \Omega,
\]
\[
|f(x, t)| \leq C|t|^{p-1} \quad \text{for every } t \in \mathbb{R}, \text{ a.e. } x \in \Omega,
\]
where \(u_\pm := \max\{\pm u, 0\}\) and \(C\) is a positive constant.

The hypotheses \((A)\) are considered in the study of the quasilinear elliptic problems (see [9,18,20,21,29]). For example, we can treat the operators like
\[
-\text{div } A(x, \nabla u) = -\text{div } \left(|\nabla u|^{p-2} (1 + |\nabla u|^p)^{-\frac{q-p}{p}} \nabla u\right) \quad \text{for } 1 < q \leq p < \infty,
\]
whence this stands for the usual \(p\)-Laplacian in the case of \(p = q\) (cf. [18,21]).

We see that the nonlinear term \(f\) as in \((F)\) is related to the equation
\[
-\text{div } A(x, \nabla u) = \alpha u_+^{p-1} - \beta u_-^{p-1}, \quad u \in W^{1,p}(\Omega).
\]
Let us recall the known results in the special case of \(A(x, y) = |y|^{p-2}y\) (that is, \(p\)-Laplace equation). The set of all points \((\alpha, \beta) \in \mathbb{R}^2\) for which the equation
\[
-\Delta_p u = \alpha u_+^{p-1} - \beta u_-^{p-1}, \quad u \in W^{1,p}(\Omega)
\]
has a non-trivial solution is called the Fučik spectrum of the \(p\)-Laplacian under the Neumann boundary condition (see [13,8–10] under the Dirichlet boundary condition and [2,3] for Neumann boundary condition). We denote the Fučik spectrum of \(p\)-Laplacian by \(\Theta_p\). In the case of \(\alpha = \beta = \lambda \in \mathbb{R}\), the Eq. \((5)\) reads \(-\Delta_p u = \lambda |u|^{p-2}u\). Hence \((\lambda, \lambda)\) belongs to \(\Theta_p\) if and only if \(\lambda\) is an eigenvalue of \(-\Delta_p\), i.e., there exists a non-zero weak solution \(u \in W^{1,p}(\Omega)\) to \(-\Delta_p u = \lambda |u|^{p-2}u\). It is well known that the first eigenvalue \(\mu_1 = 0\) of \(-\Delta_p\) is simple.