Existence of a ground state solution for a nonlinear scalar field equation with critical growth

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Abstract In the present paper, we establish the existence of Ground State Solutions for some class of Elliptic problems with Critical Growth in $\mathbb{R}^N$ for $N \geq 2$. Our results complete the study made in Berestycki and Lions (Arch Rat Mech Anal 82:313–346, 1983) and Berestycki, Gallouët and Kavian (C R Acad Sci Paris Ser I Math 297:307–310, 1984), in the sense that, in those papers only the subcritical growth was considered.

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1 Introduction

In [4] and [5], the authors studied the following minimization problem

$$\min \left\{ \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 : \int_{\mathbb{R}^N} G(u) = 1 \right\}, \text{ when } N \geq 3 \text{ or (1)}$$
and

\[
\min \left\{ \frac{1}{2} \int |\nabla u|^2 : \int \mathcal{G}(u) = 0 \right\}, \quad \text{if } N = 2, \tag{2}
\]

respectively, where \( \mathcal{G} : \mathbb{R} \to \mathbb{R} \) is the primitive of a function \( g : \mathbb{R} \to \mathbb{R} \) which has a subcritical growth. More precisely, in [4], the authors assume that \( g \) satisfies

\[
-\infty < \liminf_{s \to 0^+} \frac{g(s)}{s} \leq \limsup_{s \to 0^+} \frac{g(s)}{s} \leq -m < 0; \tag{g_1}
\]

\[
\limsup_{s \to +\infty} \frac{g(s)}{s^{2^*-1}} \leq 0; \tag{g_2}
\]

there exists \( \xi > 0 \) such that \( \mathcal{G}(\xi) > 0 \). \( \tag{g_3} \)

while in [5], the authors assume that \( g \) verifies \((g_1), (g_3)\), with \((g_2)\) being replaced by the following condition

\[
\limsup_{|s| \to +\infty} \frac{f(s)}{e^{\beta s^2}} = 0, \quad \forall \beta > 0. \tag{g_2'}
\]

In [4] and [5] the authors have shown that, under a convenient change of scale, the minimum of (1) or (2) gives rise to a round state solution for the problem

\[
\begin{cases}
-\Delta u = g(u) & \text{in } \mathbb{R}^N, \\
u > 0 & \text{in } \mathbb{R}^N, \\
u \in H^1(\mathbb{R}^N),
\end{cases} \tag{3}
\]

In [6] the authors show that the mountain pass level of the energy functional associated to (3) is a critical value and corresponds to the ground state found in [4,5]. In the present paper, we will complete this study by considering a class of nonlinearities with critical growth and under mild assumptions. Here, we will assume that \( g(s) = -s + f(s) \), where \( f : \mathbb{R} \to \mathbb{R} \) is a continuous function with critical growth. We thus obtain a ground state solution for the problem

\[
\begin{cases}
-\Delta u + u = f(u) & \text{on } \mathbb{R}^N, \\
u > 0 & \text{in } \mathbb{R}^N, \\
u \in H^1(\mathbb{R}^N),
\end{cases} \tag{4}
\]

by minimizing (1) or (2). By a ground state solution, we mean a solution such \( w \in H^1(\mathbb{R}^N) \) such that \( I(w) \leq I(v) \) for every nontrivial solution \( v \) of (4), where \( I : H^1(\mathbb{R}^N) \to \mathbb{R} \) denotes the energy functional associated to (4) defined by

\[
I(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) - \int_{\mathbb{R}^N} F(u).
\]