Boundary partial regularity for a class of biharmonic maps

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Abstract We consider the Dirichlet problem for biharmonic maps $u$ from a bounded, smooth domain $\Omega \subset \mathbb{R}^n (n \geq 5)$ to a compact, smooth Riemannian manifold $N \subset \mathbb{R}^l$ without boundary. For any smooth boundary data, we show that if $u$ is a stationary biharmonic map that satisfies a certain boundary monotonicity inequality, then there exists a closed subset $\Sigma \subset \overline{\Omega}$, with $H^{n-4}(\Sigma) = 0$, such that $u \in C^\infty(\Omega \setminus \Sigma, N)$.

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1 Introduction

This is a continuation of our previous study in [14]. Here we consider the Dirichlet problem for (extrinsic) biharmonic maps into Riemannian manifolds in dimension at least 5 and address the issue of boundary regularity for a class of stationary biharmonic maps.

For $n \geq 5$, let $\Omega \subset \mathbb{R}^n$ be a bounded, smooth domain and $(N, h) \subset \mathbb{R}^l$ be a $l$-dimensional, compact $C^3$-Riemannian manifold with $\partial N = \emptyset$. For $k \geq 1$, $1 \leq p < +\infty$, define the Sobolev space

$$W^{k,p}(\Omega, N) = \left\{ v \in W^{k,p}(\Omega, \mathbb{R}^l) : v(x) \in N \text{ for a.e.} x \in \Omega \right\}.$$

Recall that an extrinsic biharmonic map $u \in W^{2,2}(\Omega, N)$ is defined to be a critical point of the Hessian energy functional:

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\[ E_2(v) = \int_{\Omega} |\nabla v|^2, \quad v \in W^{2,2}(\Omega, N). \]

If we denote by \( P(y) : \mathbb{R}^l \to T_y N, \ y \in N, \) the orthogonal projection map, then the second fundamental form of \( N \) is defined by
\[
\mathbb{B}(y)(X, Y) = -D_x P(y)(Y), \quad \forall X, Y \in T_y N.
\]
It is standard (cf. [21] Proposition 2.1) that an extrinsic biharmonic map \( u \in W^{2,2}(\Omega, N) \) is a weak solution to the biharmonic map equation:
\[
\Delta^2 u = \Delta(\mathbb{B}(u)(\nabla u, \nabla u)) + 2 \nabla \cdot (\Delta u, \nabla (P(u))) - \langle \Delta(P(u)), \Delta u \rangle,
\]
or equivalently
\[
\Delta^2 u \perp T_u N. \tag{1.2}
\]

Notice that the biharmonic map equation (1.1) is a 4th order elliptic system with supercritical nonlinearity. It is a very natural and interesting question to study its regularity. The study was first initiated by Chang, Wang, and Yang [5]. In particular, they proved that when \( N = S^{l-1} \subset \mathbb{R}^l \) is the unit sphere, then any \( W^{2,2} \)-biharmonic map is smooth in dimension 4, and smooth away from a closed set of \((n - 4)\)-dimensional Hausdorff measure zero for \( n \geq 5 \) provided that it is, in addition, stationary. The main theorem in [5] was subsequently extended to any smooth Riemannian manifold \( N \) by the third author in [20–22], and different proofs were later given by Lamm–Rivière [13] for \( n = 4 \) and Struwe [18] for \( n \geq 5 \), see also Strzelecki [19] for some generalizations.

There have been many important studies on the regularity of stationary harmonic maps, originally due to Hélein [10], Evans [7], and Bethuel [4] (see Rivière [16], and Rivière–Struwe [17] for important new approaches and improvements). A crucial property for stationary harmonic maps is the well-known energy monotonicity formula (see Price [15]). The notion of stationary biharmonic maps was motivated by the notion of stationary harmonic maps. More precisely, \( u \in W^{2,2}(\Omega, N) \) is called a stationary biharmonic map if it is, in addition, a critical point with respect to the domain variations:
\[
\frac{d}{dt} \big|_{t=0} \int_{\Omega} |\nabla u_t|^2 = 0, \quad u_t(x) = u(x + t Y(x)), \ Y \in C_0^{\infty}(\Omega, \mathbb{R}^n). \tag{1.3}
\]

It has been derived by [5] and Angelsberg [2] that stationary biharmonic maps enjoy the following interior monotonicity inequality: for \( x \in \Omega \) and \( 0 < r \leq R < \text{dist}(x, \partial \Omega) \),
\[
R^{4-n} \int_{B_R(x)} |\Delta u|^2 - r^{4-n} \int_{B_r(x)} |\Delta u|^2 = A_1 + A_2, \tag{1.4}
\]
where
\[
A_1 = 4 \int_{B_R(x) \setminus B_r(x)} \left( \frac{|u_i + (y - x)^i u_{ij}|^2}{|y - x|^{n-2}} + (n - 2) \frac{|(y - x)^i u_{ij}|^2}{|y - x|^n} \right),
\]
\[
A_2 = 2 \int_{\partial(B_R(x) \setminus B_r(x))} \left( \frac{(y - x)^i u_{ij} u_{ij}}{|y - x|^{n-3}} + 2 \frac{|(y - x)^i u_{ij}|^2}{|y - x|^{n-1}} - 2 \frac{\nabla u^2}{|y - x|^{n-3}} \right).
\]