On the long-time behavior of some mathematical models for nematic liquid crystals

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Abstract A model describing the evolution of a liquid crystal substance in the nematic phase is investigated in terms of two basic state variables: the velocity field $u$ and the director field $d$, representing the preferred orientation of molecules in a neighborhood of any point in a reference domain. After recalling a known existence result, we investigate the long-time behavior of weak solutions. In particular, we show that any solution trajectory admits a non-empty $\omega$-limit set containing only stationary solutions. Moreover, we give a number of sufficient conditions in order that the $\omega$-limit set contains a single point. Our approach improves and generalizes existing results on the same problem.

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1 Introduction

In this paper we analyze the long-time behavior of weak solutions to the system

\[ u_t + \text{div}(u \otimes u) - \nu \Delta u = \text{div}(-pI - L(\nabla d \otimes \nabla d) - \delta(L \Delta d - f(d)) \otimes d), \]

\[ \text{div} u = 0, \]

\[ d_t + u \cdot \nabla d - \delta d \cdot \nabla u - L \Delta d + f(d) = 0, \]

describing the evolutionary behavior of nematic liquid crystal flows (we refer to the monographs [5,6] for a detailed presentation of the physical foundations of continuum theories of liquid crystals). Actually, system (1–3) can be seen as a simplification of the original Ericksen–Leslie model [7,13], that still keeps a good level of compliance with experimental results. The model couples the Navier-Stokes Eq. 1 for the macroscopic velocity \( u \) (\( p \) denoting as usual the pressure), with the incompressibility condition (2) and with the Eq. 3 ruling the behavior of the local orientation vector \( d \) of the liquid crystal. Here, the function \( f \) represents the gradient w.r.t. \( d \) of the configuration energy \( F \) of the crystal. We choose \( F \) to be a double well potential having minima for \(|d| = 1 \) and growing at infinity at most as a fourth order polynomial. This provides a standard relaxation of the physical constraint \(|d| = 1 \), which is very difficult to treat mathematically.

In this paper, the system is complemented with the homogeneous Dirichlet boundary condition for \( u \), the no-flux condition for \( d \), and with initial conditions. It is settled in a smooth bounded domain \( \Omega \subset \mathbb{R}^d \) for \( d = 2 \) or \( d = 3 \). No restriction is assumed on the viscosity coefficient \( \nu \).

Regarding the parameter \( \delta \), we will take \( \delta \geq 0 \), with the case \( \delta > 0 \) denoting the presence of a stretching effect on the molecules of the crystal. Some of our results, however, hold only for \( \delta = 0 \). Actually, the situation \( \delta > 0 \) is more difficult to be treated mathematically since the term \( \delta d \cdot \nabla u \) prevents from using maximum principle arguments in (3). For this reason, even if the initial datum \( d_0 \) satisfies the (relaxed) physical constraint \(|d_0| \leq 1 \) almost everywhere, the same may not be true for \( d(t) \), for positive times, if \( \delta > 0 \).

A mathematical analysis of system (1–3) has been first addressed in the papers [14] and [15] (in this second work, an even more general model is taken into account). There, the authors consider the case \( \delta = 0 \) and prove existence of a unique classical solution for \( d = 2 \), and also in dimension \( d = 3 \) under the additional assumption that the viscosity \( \nu \) is sufficiently large. These results have been extended to the case \( \delta > 0 \) in the paper [19]. Finally, the restriction on the viscosity has been recently dropped in [2], where weak solutions are considered and a global existence result for the 3D system (1–3) is proved in that regularity frame. Of course, uniqueness is not known to hold in that regularity setting. A similar result is essentially contained also in the recent paper [3], where analogous estimates are derived but no formal statement of an existence result is provided.

The Dirichlet boundary condition for \( u \) and either a nonhomogeneous Dirichlet or the no-flux boundary condition for \( d \) are treated there. Moreover, let us quote the recent paper [9], where these results have been extended to a more general system (1–3), where also temperature effects are taken into account. We note, however, that the results of [9] require different boundary conditions for \( u \) (namely, the so-called complete slip conditions).

The long-time behavior of system (1–3) has been analyzed in the recent work [20], still considering the case \( d = 2 \) or the case \( d = 3 \) with the large viscosity \( \nu \), and periodic boundary conditions. More precisely, in [20] the authors show existence of a nonempty \( \omega \)-limit set for any strong bounded solution emanating from smooth initial data. Moreover, by using the