Lower semicontinuity and relaxation of signed functionals with linear growth in the context of $\mathcal{A}$-quasiconvexity

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Abstract A lower semicontinuity and relaxation result with respect to weak-$\ast$ convergence of measures is derived for functionals of the form

$$\mu \in \mathcal{M}(\Omega; \mathbb{R}^d) \rightarrow \int_{\Omega} f(\mu^a(x)) \, dx + \int_{\Omega} f^\infty \left( \frac{d\mu^s}{d|\mu^s|} (x) \right) \, d|\mu^s|(x),$$

where admissible sequences $\{\mu_n\}$ are such that $\{\mathcal{A}\mu_n\}$ converges to zero strongly in $W_{loc}^{-1,q}(\Omega)$ and $\mathcal{A}$ is a partial differential operator with constant rank. The integrand $f$ has linear growth and $L^\infty$-bounds from below are not assumed.

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1 Introduction

In this work we start by deriving a lower semicontinuity result with respect to weak-$\ast$ convergence of $\mathcal{A}$-free measures for the functional

$$\mathcal{F}(\mu) = \int_{\Omega} f(\mu^a) \, dx + \int_{\Omega} f^\infty \left( \frac{d\mu^s}{d|\mu^s|} \right) \, d|\mu^s|, \quad \mu \in \mathcal{M}(\Omega; \mathbb{R}^d),$$

(1.1)

where $\Omega$ is an open bounded subset of $\mathbb{R}^N$, $\mathcal{M}(\Omega; \mathbb{R}^d)$ stands for the set of finite $\mathbb{R}^d$-valued Radon measures over $\Omega$, $\mu = \mu^a L^N + \mu^s$ is the Radon–Nikodým decomposition of $\mu$ with

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respect to the Lebesgue measure $\mathcal{L}^N$. Here and in what follows, the integrand $f : \mathbb{R}^d \to \mathbb{R}$ is assumed to be $\mathcal{A}$-quasiconvex (see Sect. 2 for other notations and definitions), where $\mathcal{A}$ is a linear first order partial differential operator of the form

$$\mathcal{A} := \sum_{i=1}^{N} A^{(i)} \frac{\partial}{\partial x_i}, \quad A^{(i)} \in M^{M \times d}(\mathbb{R}), \quad M \in \mathbb{N},$$

that we assume throughout Murat’s condition of constant rank (see [15] and [9]) i.e., there exists $c \in \mathbb{N}$ such that

$$\text{rank} \left( \sum_{i=1}^{N} A^{(i)} \xi^i \right) = c \quad \text{for all } \xi = (\xi_1, ..., \xi_N) \in S^{N-1}.$$

In addition we assume $f$ to be Lipschitz continuous and we remark that this condition implies $f$ to satisfy a linear growth condition at infinity of the type

$$|f(v)| \leq C(1 + |v|) \quad (1.3)$$

for all $v \in \mathbb{R}^d$ and for some $C > 0$. As usual we denote by $f^\infty$ the recession function of $f$ (see Remark 3.2 below) which for our problem is defined as

$$f^\infty(\xi) := \limsup_{t \to \infty} \frac{f(t \xi)}{t}. \quad (1.4)$$

As already proved by Fonseca and Müller [9] $\mathcal{A}$-quasiconvexity with respect to the last variable turns out to be a necessary and sufficient condition for the lower semicontinuity of

$$(u, v) \to \int_{\Omega} f(x, u(x), v(x)) \, dx$$

for positive normal integrands $f$ with linear growth among sequences $(u_n, v_n)$ such that $u_n \to u$ in measure, $v_n \rightharpoonup v$ in $L^1$ and $\mathcal{A} v_n = 0$. In Fonseca, Leoni and Müller [10] this result was partially extended by considering weak-$\ast$ convergence in the sense of measures (in the variable $v$). Precisely the authors considered a functional of the form

$$v \to \int_{\Omega} f(x, v(x)) \, dx$$

and, in particular, it was proved that

$$\int_{\Omega} f(x, \mu^G(x)) \, dx \leq \lim_{n \to \infty} \int_{\Omega} f(x, v_n(x)) \, dx \quad (1.5)$$

for any sequence $v_n \subset L^1(\Omega; \mathbb{R}^d) \cap \ker \mathcal{A}$ such that $v_n \mathcal{L}^N \rightharpoonup^* \mu$ in the sense of measures, under the assumptions that $f$ is a Borel measurable positive function with linear growth, Lipschitz continuous and $\mathcal{A}$-quasiconvex in the last variable, and satisfying an appropriate continuity condition on the first variable (see Theorem 1.4 in [10]). Note that in (1.5) the term $\mu^G$ has not been considered.

Here we extend this last result for a larger class of integrands where $L^\infty$-bounds from below are not assumed and to functionals taking into account the singular part of the limit measure $\mu$. Namely, we prove the following theorem.

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