Küneth formula in Rabinowitz Floer homology

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Abstract Rabinowitz Floer homology has been investigated on submanifolds of contact type. The contact condition, however, is quite restrictive. For example, a product of contact hypersurfaces is rarely of contact type. In this article, we study Rabinowitz Floer homology for product manifolds which are not necessarily of contact type. We show for a class of product manifolds that there are infinitely many leafwise intersection points by proving the Küneth formula for Rabinowitz Floer homology.

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1 Introduction

Rabinowitz Floer homology has been extensively studied in recent years because of its interrelation with the leafwise intersection problem. However Rabinowitz Floer homology (to be honest, the perturbed Rabinowitz action functional) has worked principally on a contact submanifold and little research has been conducted on a non-contact case. Our primary objective in this paper is to find leafwise intersection points and define Rabinowitz Floer homology for this class of submanifolds which are not necessarily of contact type. In addition we show for the class that there are infinitely many leafwise intersection points by proving the Küneth formula for Rabinowitz Floer homology. For simplicity, throughout this paper, we use $\mathbb{Z}/2$-coefficients for Rabinowitz Floer homology, but expect the Küneth formula continues to hold with $\mathbb{Z}$-coefficient.

We consider restricted contact hypersurfaces $(\Sigma_1, \lambda_1)$ resp. $(\Sigma_2, \lambda_2)$ in exact symplectic manifolds $(M_1, \omega_1 = d\lambda_1)$ resp. $(M_2, \omega_2 = d\lambda_2)$. Moreover we assume that $\Sigma_1$ resp. $\Sigma_2$ bounds a compact region in $M_1$ resp. $M_2$ and that those $M_1$ and $M_2$ are convex at infinity;

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that is, they are symplectomorphic to the symplectization of a compact contact manifold at infinity. Given \( F_1 \in (S^1 \times M_1), F_2 \in C^\infty(S^1 \times M_2) \), the operation
\[
(F_1 \oplus F_2)(t, x, y) = F_1(t, x) + F_2(t, y), \quad (t, x, y) \in S^1 \times M_1 \times M_2
\]
provides a time-dependent Hamiltonian function \( F_1 \oplus F_2 \in C^\infty(S^1 \times M_1 \times M_2) \). We also introduce projection maps \( \pi_1 : M_1 \times M_2 \to M_1 \) and \( \pi_2 : M_1 \times M_2 \to M_2 \); then \((M_1 \times M_2, \omega_1 \oplus \omega_2)\) admits the symplectic structure \( \omega_1 \oplus \omega_2 = \pi_1^*\omega_1 + \pi_2^*\omega_2 \).

On \((\Sigma_1 \times \Sigma_2, M_1 \times M_2)\), we define the perturbed Rabinowitz action functional \( A^\tilde{F}_1,\tilde{F}_2 \) as in (2.1). Since \( \Sigma_1 \times \Sigma_2 \) is a stable submanifold, we can define Floer homology of \( A^\tilde{F}_1,\tilde{F}_2 \) when \( F \equiv 0 \) (refer to [24] for definitions and constructions). This Floer homology \( HF(A^\tilde{F}_1,\tilde{F}_2) \) is called Rabinowitz Floer homology and denoted by \( RFH(\Sigma_1 \times \Sigma_2, M_1 \times M_2) \), see Sect. 3. By the standard continuation method in Floer theory, \( HF(A^\tilde{F}_1,\tilde{F}_2) \) and \( RFH(\Sigma_1 \times \Sigma_2, M_1 \times M_2) \) are isomorphic whenever \( HF(A^\tilde{F}_1,\tilde{F}_2) \) is defined.

**Theorem A** The Floer homologies \( RFH(\Sigma_1 \times \Sigma_2, M_1 \times M_2) \) and \( HF(A^\tilde{F}_1,\tilde{F}_2) \) are well-defined. Moreover, we have the following Künneth formula in Rabinowitz Floer homology:
\[
RFH_n(\Sigma_1 \times \Sigma_2, M_1 \times M_2) \cong \bigoplus_{p=0}^{n} RFH_p(\Sigma_1, M_1) \otimes RFH_{n-p}(\Sigma_2, M_2).
\]

Here, \( RFH_p(\Sigma_1, M_1) \) (resp. \( RFH_{n-p}(\Sigma_2, M_2) \)) is the Rabinowitz Floer homology for the restricted contact hypersurface \( \Sigma_1 \) in \( M_1 \) (resp. \( \Sigma_2 \) in \( M_2 \)), see [1] or Sect. 3.

**Remark 1.1** In this paper, we unfortunately establish compactness of gradient flow lines of the Rabinowitz action functional only for perturbations of the form \( F = F_1 \oplus F_2 \). Thus we cannot study the existence problem of leafwise intersection points for an arbitrary perturbation. However, if \( \Sigma_1 \times \Sigma_2 \) has contact type in the sense of Bolle [10,11] (see Sect. 4), the Floer homology \( HF(A^\tilde{F}_1,\tilde{F}_2) \) is defined for all perturbations, see [24]. We note that, in general, \( \Sigma_1 \times \Sigma_2 \) is not of contact type in the sense of Bolle. For example, \( S^3 \times S^3 \) is not a contact submanifold in \( \mathbb{R}^8 \), see Remark 4.2.

**Question 1.2** What perturbations have a leafwise intersection point on \((\Sigma_1 \times \Sigma_2, M_1 \times M_2)\)?

**Remark 1.3** Once one verifies compactness of gradient flow lines of the Rabinowitz action functional for a given perturbation \( F \), it guarantees the existence of leafwise intersection points for that \( F \) by using the stretching the neck argument in [1]. In this paper, we are able to compactify gradient flow lines of \( A^\tilde{F}_1,\tilde{F}_2 \), and thus it guarantees the existence of leafwise intersection points of \( F_1 \oplus F_2 \); but, this directly follows from the result in [1] that each \( F_1 \) and \( F_2 \) has a leafwise intersection point on \( \Sigma_1 \) and \( \Sigma_2 \) respectively.

**Definition 1.4** The Hamiltonian vector field \( X_F \) on a symplectic manifold \((M, \omega)\) is defined explicitly by \( i_{X_F}\omega = dF \) for a Hamiltonian function \( F \in C^\infty(S^1 \times M) \), and we call its time one flow \( \phi_F \) the Hamiltonian diffeomorphism. We denote by \( \text{Ham}(M, \omega) \) the group of Hamiltonian diffeomorphisms generated by compactly supported Hamiltonian function. This group has a well-known norm introduced by Hofer (see Definition 2.2).

**Definition 1.5** We denote by \( \varphi(\Sigma_1, \lambda_1) > 0 \) the minimal period of closed Reeb orbits of \((\Sigma_1, \lambda_1)\) which are contractible in \( M_1 \). If there is no contractible closed Reeb orbit we set \( \varphi(\Sigma_1, \lambda_1) = \infty \).

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