Four positive solutions for the semilinear elliptic equation: $-\Delta u + u = a(x)u^p + f(x)$ in $\mathbb{R}^N$

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Received December 11, 1998 / Accepted July 16, 1999 / Published online April 6, 2000 – © Springer-Verlag 2000

Abstract. We consider the existence of positive solutions of the following semilinear elliptic problem in $\mathbb{R}^N$:

$$-\Delta u + u = a(x)u^p + f(x) \quad \text{in } \mathbb{R}^N,$$

$$u > 0 \quad \text{in } \mathbb{R}^N,$$

$$u \in H^1(\mathbb{R}^N),$$

where $1 < p < \frac{N+2}{N-2} (N \geq 3)$, $1 < p < \infty (N = 1, 2)$, $a(x) \in C(\mathbb{R}^N)$, $f(x) \in H^{-1}(\mathbb{R}^N)$ and $f(x) \geq 0$. Under the conditions:

1° $a(x) \in (0, 1]$ for all $x \in \mathbb{R}^N$,

2° $a(x) \to 1$ as $|x| \to \infty$,

3° there exist $\delta > 0$ and $C > 0$ such that

$$a(x) - 1 \geq -Ce^{-(2+\delta)|x|} \quad \text{for all } x \in \mathbb{R}^N,$$

4° $a(x) \not\equiv 1$,

we show that (*) has at least four positive solutions for sufficiently small $\|f\|_{H^{-1}(\mathbb{R}^N)}$ but $f \not\equiv 0$.

Mathematics Subject Classification (1991): 35J20

* Partially supported by Waseda University Grant for Special Research Projects 98A-122, 99A-190.
0. Introduction

In this paper, we study the existence and the multiplicity of positive solutions for the following semilinear elliptic problem:

\[-\Delta u + u = a(x)u^p + f(x) \quad \text{in } \mathbb{R}^N,\]
\[u > 0 \quad \text{in } \mathbb{R}^N,\]
\[u \in H^1(\mathbb{R}^N),\]

(0.1) (0.2) (0.3)

where \(1 < p < \frac{N+2}{N-2}\) (\(N \geq 3\)), \(1 < p < \infty\) (\(N = 1, 2\)). We assume that \(a(x) \in C(\mathbb{R}^N)\) satisfies

\[a(x) > 0 \quad \text{for all } x \in \mathbb{R}^N,\]
\[a(x) \to 1 \quad \text{as } |x| \to \infty\]

(0.4) (0.5)

and \(f(x)\) satisfies

\[f(x) \in H^{-1}(\mathbb{R}^N),\]
\[f(x) \geq 0.\]

(0.6) (0.7)

Under the assumptions (0.4)–(0.7), our problem (0.1)–(0.3) can be regarded as a perturbation problem of the following homogeneous problem:

\[-\Delta u + u = u^p \quad \text{in } \mathbb{R}^N,\]
\[u > 0 \quad \text{in } \mathbb{R}^N,\]
\[u \in H^1(\mathbb{R}^N).\]

(0.8) (0.9) (10)

It is known that (0.8)–(0.10) has a unique positive radial solution \(\omega(x) = \omega(|x|)\) and any positive solution \(u(x)\) of (0.8)–(0.10) can be written as

\[u(x) = \omega(x - x_0) \quad \text{for some } x_0 \in \mathbb{R}^N.\]

(See Kwong [17], c.f. Kabeya-Tanaka [16]).

Our main question is whether positive solutions can survive after a perturbation of type (0.1)–(0.3) or not. Such a question was studied by Zhu [25], Cao-Zhou [11], Jeanjean [15], Hirano [14] and Adachi-Tanaka [1]. See also Ambrosetti and Badiale [3] for a perturbation result via Poincaré-Melnikov type arguments. Zhu [25] (c.f. Hirano [14]) were mainly concerned with the case \(a(x) \equiv 1\) and \(f(x) \geq 0, f(x) \not\equiv 0\) and succeeded to find the existence of at least two positive solutions under the situation

\[\|f\|_{H^{-1}(\mathbb{R}^N)} \leq M,\]

(0.11)