Four positive solutions for the semilinear elliptic equation: 
\(-\Delta u + u = a(x)u^p + f(x) \text{ in } \mathbb{R}^N\)

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Abstract. We consider the existence of positive solutions of the following semilinear elliptic problem in \(\mathbb{R}^N\):

\[-\Delta u + u = a(x)u^p + f(x) \quad \text{in } \mathbb{R}^N,\]
\[u > 0 \quad \text{in } \mathbb{R}^N,\]
\[u \in H^1(\mathbb{R}^N),\]

where \(1 < p < \frac{N+2}{N-2}\) (\(N \geq 3\)), \(1 < p < \infty\) (\(N = 1, 2\)), \(a(x) \in C(\mathbb{R}^N)\), \(f(x) \in H^{-1}(\mathbb{R}^N)\) and \(f(x) \geq 0\). Under the conditions:

1° \(a(x) \in (0, 1]\) for all \(x \in \mathbb{R}^N\),
2° \(a(x) \to 1\) as \(|x| \to \infty\),
3° there exist \(\delta > 0\) and \(C > 0\) such that

\[a(x) - 1 \geq -Ce^{-(2+\delta)|x|} \quad \text{for all } x \in \mathbb{R}^N,\]

4° \(a(x) \not\equiv 1\),

we show that (*) has at least four positive solutions for sufficiently small \(\|f\|_{H^{-1}(\mathbb{R}^N)}\) but \(f \not\equiv 0\).

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0. Introduction

In this paper, we study the existence and the multiplicity of positive solutions for the following semilinear elliptic problem:

\[-\Delta u + u = a(x)u^p + f(x) \quad \text{in } \mathbb{R}^N, \quad (0.1)\]
\[u > 0 \quad \text{in } \mathbb{R}^N, \quad (0.2)\]
\[u \in H^1(\mathbb{R}^N), \quad (0.3)\]

where $1 < p < \frac{N + 2}{N - 2}$ ($N \geq 3$), $1 < p < \infty$ ($N = 1, 2$). We assume that $a(x) \in C(\mathbb{R}^N)$ satisfies

\[a(x) > 0 \quad \text{for all } x \in \mathbb{R}^N, \quad (0.4)\]
\[a(x) \to 1 \quad \text{as } |x| \to \infty \quad (0.5)\]

and $f(x)$ satisfies

\[f(x) \in H^{-1}(\mathbb{R}^N), \quad (0.6)\]
\[f(x) \geq 0. \quad (0.7)\]

Under the assumptions (0.4)–(0.7), our problem (0.1)–(0.3) can be regarded as a perturbation problem of the following homogeneous problem:

\[-\Delta u + u = u^p \quad \text{in } \mathbb{R}^N, \quad (0.8)\]
\[u > 0 \quad \text{in } \mathbb{R}^N, \quad (0.9)\]
\[u \in H^1(\mathbb{R}^N). \quad (0.10)\]

It is known that (0.8)–(0.10) has a unique positive radial solution $\omega(x) = \omega(|x|)$ and any positive solution $u(x)$ of (0.8)–(0.10) can be written as

\[u(x) = \omega(x - x_0) \quad \text{for some } x_0 \in \mathbb{R}^N. \quad (0.11)\]

(See Kwong [17], c.f. Kabeya-Tanaka [16]).

Our main question is whether positive solutions can survive after a perturbation of type (0.1)–(0.3) or not. Such a question was studied by Zhu [25], Cao-Zhou [11], Jeanjean [15], Hirano [14] and Adachi-Tanaka [1]. See also Ambrosetti and Badiale [3] for a perturbation result via Poincaré-Melnikov type arguments. Zhu [25] (c.f. Hirano [14]) were mainly concerned with the case $a(x) \equiv 1$ and $f(x) \geq 0$, $f(x) \neq 0$ and succeeded to find the existence of at least two positive solutions under the situation

\[\|f\|_{H^{-1}(\mathbb{R}^N)} \leq M, \quad (0.11)\]