Fibers of the Baum-Bott map for foliations of degree two on $\mathbb{P}^2$

Alcides Lins Neto

— Dedicated to Jean-François Mattei in his 60th birthday

Abstract. The Baum-Bott map associates to a foliation the Baum-Bott indexes of their singularities. In this paper we study the fibers of the Baum-Bott map in the space of foliations of degree two on the projective plane $\mathbb{P}^2$. In the main result we prove that its generic fiber contains exactly 240 orbits of the natural action of $\text{Aut}(\mathbb{P}^2)$ on the space of foliations.

Keywords: holomorphic foliation, Baum-Bott map.

Mathematical subject classification: 37F75, 34M15.

1 Introduction

1.1 The Baum-Bott map

One of the most basic invariant for singularities of holomorphic foliations of surfaces is the Baum-Bott index: if $\mathcal{F}$ is a holomorphic foliation on a neighborhood $U$ of $p \in \mathbb{C}^2$, induced by a holomorphic 1-form $\omega = A(x, y) \, dy - B(x, y) \, dx$, with an unique singularity at $p$, then the Baum-Bott index of $\mathcal{F}$ at $p$ is defined as

$$BB(\mathcal{F}, p) = \frac{1}{(2\pi i)^2} \int_{\Gamma} \eta \wedge d\eta,$$

where $\eta$ is any $(1, 0)$-form, $C^\infty$ on $U \setminus \{p\}$, satisfying $d\omega = \eta \wedge \omega$, and $\Gamma$ is the boundary of a ball $B$ around $p$ with $p \in B \subset \overline{B} \subset U$ (cf. [1]). Note that if $f \in \mathcal{O}^*(U)$ and $\omega_1 = f \cdot \omega$ then $d\omega_1 = \eta_1 \wedge \omega_1$, where $\eta_1 = \eta + \frac{df}{f}$, so that

$$\eta_1 \wedge d\eta_1 = \eta \wedge d\eta + d \left( \eta_1 \wedge \frac{df}{f} \right) \quad \Longrightarrow \quad \int_{\Gamma} \eta \wedge d\eta = \int_{\Gamma} \eta_1 \wedge d\eta_1.$$

Received 23 May 2011.
In particular, the Baum-Bott index does not depend on the 1-form representing the foliation.

Another important fact is that it is invariant by biholomorphisms; if \( \varphi : (V, q) \to (U, p) \) is a biholomorphism then \( BB(\varphi^*(F), q) = BB(F, p) \) (cf. [1]).

When the dual vector field \( X = A(x, y)\partial_x + B(x, y)\partial_y \) has invertible linear part, i.e., \( \det DX(p) \neq 0 \), a simple computation shows that

\[
BB(F, p) = \frac{tr^2(DX(p))}{\det(DX(p))}.
\]

In particular, if the eigenvalues of \( DX(p) \) are \( \lambda_1 \) and \( \lambda_2 \) then

\[
BB(F, p) = \frac{(\lambda_1 + \lambda_2)^2}{\lambda_1 \lambda_2} = \frac{\lambda_2}{\lambda_1} + \frac{\lambda_1}{\lambda_2} + 2.
\]

The numbers \( \lambda_2/\lambda_1 \) and \( \lambda_1/\lambda_2 \) will be called the characteristic values of the singularity. Note that the characteristic values satisfy the equation

\[
z^2 + (2 - BB(F, p))z + 1 = 0.
\]

Singularities with invertible linear part will be called non-degenerate singularities.

In this paper, we will deal with holomorphic foliations on the complex projective plane \( \mathbb{P}^2 \). A holomorphic foliation on \( \mathbb{P}^2 \) can be defined in an affine coordinate system \( (x, y) \in \mathbb{C}^2 \subset \mathbb{P}^2 \) by a holomorphic vector field \( X = P(x, y)\partial_x + Q(x, y)\partial_y \), or by its dual 1-form \( \omega = P(x, y)dy - Q(x, y)dx \), where \( P \) and \( Q \) are polynomials. We will denote the induced foliation by \( F_X \) or \( F_\omega \). The degree of \( F_X \) is defined as the number of tangencies of the foliation and a generic line \( \ell \subset \mathbb{P}^2 \). It can be proved that if a vector field \( X = P(x, y)\partial_x + Q(x, y)\partial_y \) induces a degree \( d \) foliation then

\[
P(x, y) = p(x, y) + x g(x, y) \quad \text{and} \quad Q(x, y) = q(x, y) + y g(x, y),
\]

where \( p, q, g \in \mathbb{C}[x, y], \max(dg(p), dg(q)) \leq d \) (\( dg = \) degree) and \( g \) is homogeneous of degree \( d \). When \( g \neq 0 \) the set of directions given by \( (g(x, y) = 0) \), in the line at infinity \( L_\infty \) of \( \mathbb{C}^2 \), defines the set of tangencies of \( F_X \) with \( L_\infty \). We will denote the set of foliations of degree \( d \) on \( \mathbb{P}^2 \) by \( \mathbb{F}\text{ol}(d, 2) \) and the set of singularities of a foliation \( F \in \mathbb{F}\text{ol}(d, 2) \) by \( \text{sing}(F) \). The set of foliations of degree \( d \) and with only non-degenerate singularities will be denoted by \( \mathbb{F}\text{ol}_{nd}(d, 2) \).

\[\text{Bull Braz Math Soc, Vol. 43, N. 1, 2012}\]