New Approach to Quantum Scattering Near the Lowest Landau Threshold for a Schrödinger Operator with a Constant Magnetic Field

M. Melgaard*,**

Department of Mathematics, Chalmers University of Technology and University of Gothenburg, Eklandagatan 86, S-412 96 Gothenburg, Sweden

Received December 11, 2000; accepted in final form June 16, 2001
Published online June 10, 2002; © Springer-Verlag 2002

Abstract. For a fixed magnetic quantum number $m$ results on spectral properties and scattering theory are given for the three-dimensional Schrödinger operator with a constant magnetic field and an axisymmetrical electric potential $V$. Asymptotic expansions for the resolvent of the Hamiltonian $H_m = H_{om} + V$ are deduced as the spectral parameter tends to the lowest Landau threshold $E_0$. In particular it is shown that $E_0$ can be an eigenvalue of $H_m$. Furthermore, asymptotic expansions of the scattering matrix associated with the pair $(H_m, H_{om})$ are derived as the energy parameter tends to $E_0$.

1 Introduction

Spectral and scattering theory for the three-dimensional Schrödinger operator with a constant magnetic field

\[ H(A) = H_0(A) + V(x) = (-i \nabla - A)^2 + V(x), \quad A = (1/2)B \times x, \quad (1.1) \]

has received substantial attention due to applications in astrophysics and solid-state physics (see the survey in ref. [21] and references therein) as well as mathematical interest. The basic mathematical aspects of the scattering theory for the pair $(H(A), H_0(A))$ have been studied in ref. [5] where the existence and completeness of the corresponding wave operators were proven for a large class of potentials $V$ (see also ref. [20] for a more recent extension of these results).

* E-mail: melgaard@math.chalmers.se
** EU TMR post doc funded by grant no. ERBFMRX-CT97-0159
This work concerns problems arising in the context of near-threshold scattering for the pair \((H(A), H_0(A))\), when the energy parameter approaches the lowest Landau threshold. A lot of work has been done in this field for Schrödinger operators without external fields. Classic results going back to the late forties and early fifties treat the radial symmetric case. In the late seventies, Newton [26] was the first to give detailed results on various threshold properties of three-dimensional Schrödinger operators with local (noncentral) potentials. His work was followed by the monumental work by Jensen and Kato [14]. Based on a detailed analysis of the zero-energy properties of the three-dimensional Schrödinger operator \(-\Delta + V(x)\) with \(V\) satisfying an abstract short-range condition, Jensen and Kato deduce asymptotic expansions of the full resolvent as the spectral parameter tends to zero (the so-called low-energy limit). As an application they derive expansions of the scattering matrix as the energy parameter goes to zero. The closely related problem of coupling constant thresholds was studied by Klaus and Simon [17].

Threshold scattering in the three-dimensional case was then reconsidered in a very systematic way by Albeverio, Gesztesy and various co-workers [3, 2, 4], and in the two-dimensional case by Cheney [11] with a complete treatment provided later by Bollé, Gesztesy, and Danneels [7]. The case of nonlocal interactions in three dimensions was first considered by Newton [27] and later completely resolved by Bollé, Gesztesy, Nessmann, and Streit [9]. An excellent survey of threshold properties of Schrödinger operators in dimensions one, two, and three can be found in ref. [6].

Despite its obvious importance much less is known on such problems for the operator \(H(A)\) in Eq. (1.1), which is probably explained by the additional complications that arise (see below).

We restrict ourselves to the case, where the electric potential is axisymmetric, i.e. \(V(x) = V(\rho, z), \rho = (x^2 + y^2)^{1/2}\), and decays like \(V(x) = O(|x|^{-\alpha})\) as \(|x| \to \infty\) for some \(\alpha > 2\). Furthermore, we assume that the magnetic field has constant strength 2 and is aligned in the \(z\) direction. For fixed magnetic quantum number \(m\) the resulting Hamiltonian \(H_m = H_{om} + V\) on the Hilbert space \(\mathcal{H} = L^2(\mathbb{R}_+ \times \mathbb{R}, \rho \, dp \, dz)\) has the structure of an infinite-channel operator-valued matrix. With respect to the projection \(\mathcal{P}_0\) onto the lowest Landau level we can represent \(H_m\) in a two-channel framework

\[
H_m = \begin{pmatrix}
H_0 & 0 \\
0 & H_1
\end{pmatrix}
\begin{pmatrix}
0 & V_{01} \\
V_{10} & 0
\end{pmatrix} = \begin{pmatrix}
H_{om} & V_{01} \\
V_{10} & 0
\end{pmatrix}
\]

(1.2)

on \(\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1\), where \(\mathcal{H}_0 = \text{Ran} \mathcal{P}_0\) and \(\mathcal{H}_1\) denotes its complement. By construction, \(H_0\) and \(H_1\) are self-adjoint operators in \(\mathcal{H}_0\) and \(\mathcal{H}_1\), respectively. Moreover \(V_{01}^* = V_{10}\) and \(V_{01} \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_0)\).

Due to the diagonal structure of the uncoupled Hamiltonian \(H_{om}\), its spectrum is the union of the spectra of \(H_0\) and \(H_1\), respectively. We have \(\sigma_{ac}(H_0) = [E_0, \infty)\) and \(\sigma_{ac}(H_1) = [E_1, \infty)\), where \(E_n = 2(|m| - m + 1 + 2n), n = 0, 1, 2, \ldots\), are the Landau levels. There are several possible, mostly fairly “singular” cases to treat, e.g. the one where we assume that \(E_0\) is an isolated eigenvalue of \(H_1\). Thus, \(H_{om}\) has an eigenvalue embedded at \(E_0\); the bottom of its continuous spectrum.