Optimizing Area and Perimeter of Convex Sets for Fixed Circumradius and Inradius

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Abstract. In this paper two problems posed by Santaló are solved: we determine the planar convex sets which have maximum and minimum area or perimeter when the circumradius and the inradius are given, obtaining complete systems of inequalities for the cases \((A, R, r)\) and \((p, R, r)\).

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1. Introduction

Let \(K\) be a convex set in the plane. Associated with \(K\) are a number of well-known functionals: the area \(A = A(K)\), the perimeter \(p = p(K)\), the diameter \(D = D(K)\), the minimal width \(\omega = \omega(K)\), the inradius \(r = r(K)\) and the circumradius \(R = R(K)\). For many years mathematicians have been interested in inequalities involving these functionals; and moreover, in many cases the question arises for which convex sets the equality sign is attained, that is, to determine the extremal sets.

Each new inequality obtained is interesting on its own, but it is also possible to ask if a collection of inequalities involving several geometric magnitudes is large enough to determine the existence of the figure. Such a collection is called a complete system of inequalities: a system of inequalities relating all the geometric characteristics such that for any set of numbers satisfying those conditions, a planar figure with these values of the characteristics exists in the given class.

In 1961, Santaló [6] studied complete systems of inequalities concerning triples of the six classic geometric measures: he asked for a characterization of the set of all points in \(E^3\) of the form \((a_1(K), a_2(K), a_3(K))\), where \(a_i, i = 1, 2, 3\) represent three of the six classic geometric quantities, as \(K\) ranges over the family of all compact convex sets in \(E^2\). Following an approach by Blaschke [1], Santaló

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proposed mapping the family of compact planar convex sets into a compact region of the unit square \([0,1] \times [0,1] \subset E^2\), which is called the Santaló Diagram (see Section 5). The extremal sets for the considered inequalities were mapped into the boundary points of this diagram.

The solution is trivial for subsets consisting of a single quantity. Suitable known inequalities form a complete system of inequalities for each pair of these quantities, as observed by Santaló [6], who also provided the solutions for \((A, p, \omega), (A, p, r), (A, p, R), (A, D, \omega), (p, D, \omega), \text{ and } (D, r, R)\). He left the remaining cases as open problems. Recently, in [3], [4] and [5], the cases \((D, \omega, R), (\omega, R, r), (D, \omega, r), (A, D, R)\) and \((p, D, R)\) have been settled by the second author and Segura Gomis.

In this paper, we derive four new inequalities relating the area or the perimeter with the circumradius and the inradius of a planar convex set (Theorems 1 and 2). More precisely, we determine the sets with maximum and minimum area or perimeter for fixed circumradius and inradius. Then, we will use these results to obtain the complete systems of inequalities for the cases \((A, R, r)\) and \((p, R, r)\), determining their corresponding Santaló Diagrams.

## 2. Results

For the sake of brevity, let us denote by \(B^2(\rho)\) the disc centered in the origin of coordinates \(O\) and with radius \(\rho\).

For the area, the circumradius and the inradius of a planar convex set \(K\), the well-known relationships between pairs of these geometric measures are (see, for instance, [2])

\[
A \leq \pi R^2 \quad \text{Equality for the circle} \tag{1}
\]

\[
A \geq \pi r^2 \quad \text{Equality for the circle} \tag{2}
\]

\[
r \leq R \quad \text{Equality for the circle} \tag{3}
\]

Now, for the perimeter, the circumradius and the inradius of \(K\), the well-known relationships between pairs of these geometric measures are (3) and

\[
p \leq 2\pi R \quad \text{Equality for the circle} \tag{4}
\]

\[
p \geq 4R \quad \text{Equality for the line segment} \tag{5}
\]

\[
p \geq 2\pi r \quad \text{Equality for the circle} \tag{6}
\]

(see also [2]).

But in both cases \((A, R, r)\) and \((p, R, r)\), no inequality relating the three measures is known. We prove the following theorems.

**Theorem 1.** Let \(K\) be a compact convex set in the euclidean plane \(E^2\). Then,

\[
A \leq 2 \left( r\sqrt{R^2 - r^2} + R^2 \arcsin \frac{r}{R} \right) \tag{7}
\]