On Exponential Sums Involving the Ideal Counting Function in Quadratic Number Fields

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Abstract. Let \( \chi \) be a Dirichlet character modulo \( k > 1 \), and \( F_{\chi}(n) \) the arithmetical function which is generated by the product of the Riemann zeta-function and the Dirichlet \( L \)-function corresponding to \( \chi \) in \( \Re(s) > 1 \). In this paper we study the asymptotic behaviour of the exponential sums involving the arithmetical function \( F_{\chi}(n) \). In particular, we study summation formulas for these exponential sums and mean square formulas for the error term.

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1. Introduction and Main Results

Let \( K \) be a quadratic number field with discriminant \( D \), and \( F(n) \) the number of integral ideals of \( K \) with norm equal to \( n \). It is well-known that

\[
F(n) = \sum_{d \mid n} \chi(d)
\]

with a real primitive Dirichlet character \( \chi \) modulo \( |D| \). For instance, the function \( 4^{-1}r(n) \), where \( r(n) \) is the number of integer solutions of the Diophantine equation \( n_1^2 + n_2^2 = n \), is the ideal counting function of the Gaussian number field \( \mathbb{Q}(\sqrt{-1}) \). The properties of \( r(n) \) have been investigated by many authors. In particular, the sum \( \sum_{n \leq x} r(n) \) is essentially equivalent to the number of lattice-points in the plane circle with the center at the origin and the radius \( \sqrt{x} \). It is well-known that this sum is approximated by its area \( (= \pi x) \), and one of the most remarkable problems in analytic number theory, which is called “the Gaussian circle problem”, is to find the best possible estimate of the error term related to this approximation. See [11, Chapter 3] and [6, Chapters 3 and 13] for topics on this problem.

For general \( |D| \), the average order of \( F(n) \) was obtained by Dirichlet, who showed that

\[
\sum_{n \leq x} F(n) = L(1, \chi)x + O(|D|x^{1/2}),
\]

(1.1)

where \( L(s, \chi) \) denotes the Dirichlet \( L \)-function corresponding to \( \chi \). The error term in the above asymptotic formula can be improved with respect to \( x \). However, we
shall discuss this improvement in a more general setting. Let
\[ F_\chi(n) = \sum_{d|n} \chi(d), \]
where \( \chi \) is an arbitrary primitive Dirichlet character modulo \( k > 1 \), and consider the summatory function of \( F_\chi(n) \) instead of \( F(n) \). An asymptotic formula similar to (1.1) can be shown for this sum, and the best estimate of the error term with respect to \( x \) at present is due to Huxley and Watt [5]. They proved the formula
\[
\sum_{n \leq x} F_\chi(n) = L(1, \chi)x + O\left(k^{50/73} x^{23/73} (\log x)^{461/146}\right)
\]
for \( x \geq A k \) with a certain positive constant \( A \), and they further proved that if \( x \geq A k^B \) with a sufficiently large \( B > 0 \), the error term in (1.2) can be replaced by
\[
O\left(d(k)^{4/73} k^{46/73} x^{23/73} (\log x)^{461/146}\right).
\]
Here \( d(n) \) is the divisor function.

In this paper we shall study the exponential sum involving the arithmetical function \( F_\chi(n) \). Let \( h \) and \( q \) be co-prime integers with \( q \geq 1 \), and \( e(\alpha) = \exp(2\pi i \alpha) \) for \( \alpha \in \mathbb{R} \). We put
\[
R(x; h/q) = \sum_{n \leq x} F_\chi(n)e(hn/q),
\]
where the symbol \( \sum' \) indicates that the last term is to be halved if \( x \) is an integer. In [3], [7] and [8], the case \( k = 1 \), namely \( F_\chi(n) = d(n) \), was investigated. In [7] and [8], Jutila investigated several types of Voronoi formulas for the error term related to \( R(x; h/q) \) for \( k = 1 \) and the mean square of the error term. In [3], the author studied the mean square of the error term more closely. In particular, he improved the estimate of the error term in the mean square formula with respect to \( x \) and proved an omega estimate of this error term when \( h \) and \( q \) are fixed integers.

We shall generalize those results to the case of general \( k > 1 \). Let \( F_\chi(s; h/q) \) be the generating function of \( F_\chi(n)e(hn/q) \), which is
\[
F_\chi(s; h/q) = \sum_{n=1}^{\infty} F_\chi(n)e(hn/q)n^{-s} \quad (\sigma > 1).
\]
The analytic properties of \( F_\chi(s; h/q) \) were studied by Müller [14, Lemma 1]. In particular, the function \( F_\chi(s; h/q) \) is capable of analytic continuation as an entire function if \( 1 < (k, q) < k \), and a meromorphic function which is holomorphic in the whole complex plane except for a simple pole at \( s = 1 \) if \( (k, q) = 1 \) or \( k \) (see also Lemma 1 below).

Let \( P(x; h/q) \) be the remainder term defined by
\[
P(x; h/q) = R(x; h/q) - x \text{Res}_{s=1} F_\chi(s; h/q) - F_\chi(0; h/q).
\]
The purpose of this paper is to study the asymptotic behaviour of \( P(x; h/q) \). In particular, we derive a non-trivial upper bound and mean square formulas for \( P(x; h/q) \).