Spin Groups Over a Commutative Ring
and the Associated Root Data

By

Hisatoshi Ikai

Tohoku University, Sendai, Japan

Received November 6, 2001; in revised form September 5, 2002
Published online May 9, 2003 © Springer-Verlag 2003

Abstract. Usual constructions of spin and Clifford groups, restricted to the case where the basic quadratic spaces is hyperbolic, are examined as those of group schemes over a commutative ring. Not surprisingly they yield indeed smooth and reductive group schemes, but the proof involves a lot of calculation and uses a connection with Jordan pairs. Maximal tori and their associated root data are also constructed explicitly.

2000 Mathematics Subject Classification: 14L15, 15A75, 17C30
Key words: Spin groups, exterior algebras, jordan pairs, root data

Introduction

The construction of spin groups by means of Clifford algebras goes far back to Lipschitz [10], and has been refined in functorial fashion by Chevalley [4] over fields, and by Bass [1] over commutative rings. As well as other classical groups, it is now just the same with spin groups that the usual definitions may be read scheme-theoretically. However, on account of the Demazure-Grothendieck theory [6] of group schemes, mere redefinitions are not quite satisfactory as substantial examples. Algebro-geometrical properties, like smoothness, are usually presupposed in crucial definitions, and their actual verifications are not necessarily trivial. The situation is somehow moderate for ordinary classical groups, but rather delicate for spin groups. For example, the Jacobian criterion [7, II, §5, 2.7] might be inappropriate for proving spin groups to be smooth, since their defining equations written naively soon overflow.

In the present article, we are exclusively concerned with the spin groups of hyperbolic quadratic modules. We shall prove that they are indeed smooth and reductive group schemes, and after localizing construct déploiments together with the associated root data, conforming to Theorem of Chevalley-Demazure [6, Vol. 3]. The restriction to the hyperbolic case settles our groups in connection with Jordan pairs; a group version of the Tits-Kantor-Koecher construction, due to Loos [12], being now applicable. By means of our spin groups, we shall work out new examples for [12] also. Necessary efforts in fact largely overlap with those required by [6]. We follow [12] to imbed a Jordan pair, essentially that of
alternating matrices, into our groups and obtain smooth and open neighborhoods of unit sections (3.7); this makes almost clear that our groups are smooth and how déploiements should be. Furthermore we describe, with aid of a nontrivial step of calculation (2.6), the induced group germ structures on these neighborhoods, and this in turn promises a direct proof of reductivities (3.12).

In practice we treat the hyperbolic quadratic module \( H(M) \) of an arbitrary finitely generated projective module \( M \), and take the underlying module of the exterior algebra \( \bigwedge(M) \) as the space of spinors. Certain complicatednesses that spin groups bear by nature are thus concentrated on \( \bigwedge(M) \), and we confront them in the first two sections (Sections one and two) before taking up group schemes regularly (Section three). Furthermore, as in the usual picture of construction, the spin group is enlarged to the special Clifford group and equipped with the projection onto the special orthogonal group. After confirming that they are smooth and reductive group schemes, we construct their maximal tori under the setup where \( M \) is free with a base \( e_1, e_2, \ldots, e_m \) (Section four); the root data being the expected type \( D_m \) and, in particular, the simply connected one for the spin group.

Throughout we work over an arbitrary commutative base ring \( k \). We follow mostly the conventions of [7], with some abuse of notation of Grothendieck. A finitely generated projective \( k \)-module \( E \) will be tacitly confused with the vector bundle \( W(E) \) over \( \text{Spec} \ k \), which we understand the functor sending any scalar extension \( k \to R \) to the \( R \)-module \( E \otimes_k R \); we have \( W(E) = E \otimes_k k \), in the notation of [7] and of Grothendieck. A polynomial map from \( E \) to another finitely generated projective \( k \)-module \( E_0 \) means a morphism \( W(E) \to W(E_0) \) of \( k \)-schemes, and the terminology dense is understood as universally scheme-theoretical dense [6, Exp. XVIII].

1. Preliminaries on Exterior Algebras

1.1 Basic duality. Let \( M \) be a finitely generated projective \( k \)-module. We denote by \( M^* \) the \( k \)-module \( \text{Hom}(M, k) \) dual to \( M \), and by \( \langle \cdot, \cdot \rangle : M \times M^* \to k \) the evaluation pairing. The exterior algebra \( \bigwedge(M) \) is always understood to be graded

\[
\bigwedge(M) = \bigoplus_{p \geq 0} \bigwedge^p(M) = \bigwedge^+(M) \bigoplus \bigwedge^-(M)
\]

by the ordinary degrees and by the parities of them. Similar conventions apply to \( \bigwedge(M^*) \) also. Among natural extensions apply to \( \bigwedge(M^*) \), the bilinear map \( \langle \cdot, \cdot \rangle : \bigwedge(M) \times \bigwedge(M^*) \to k \), where the vanishingness on every \( \bigwedge^p(M) \times \bigwedge^q(M^*) \) with \( p \neq q \) and the property \( \langle 1, 1 \rangle = 1 \) are presupposed, we choose once and for all the one such that, for any \( p \geq 1 \), \( x_1, \ldots, x_p \in M \), \( f_1, \ldots, f_p \in M^* \),

\[
\langle x_1 \wedge \cdots \wedge x_p, f_1 \wedge \cdots \wedge f_p \rangle = (-1)^{p(p-1)/2} \det(\langle x_i, f_j \rangle).
\]  

(1.1.1)

Since \( M \) is supposed to be finitely generated and projective, this indeed settles \( \bigwedge(M) \) (resp. \( \bigwedge^p(M) \)) and \( \bigwedge(M^*) \) (resp. \( \bigwedge^p(M^*) \)) to be dual to each other [2, III, §11.5].