A Remark on the Growth of the Denominators of Convergents

By

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Abstract. For \( \log \frac{1+\sqrt{5}}{2} \leq \lambda_* \leq \lambda^* < \infty \), let \( E(\lambda_*, \lambda^*) \) be the set

\[
\left\{ x \in [0, 1) : \liminf_{n \to \infty} \frac{\log q_n(x)}{n} = \lambda_*, \quad \limsup_{n \to \infty} \frac{\log q_n(x)}{n} = \lambda^* \right\}.
\]

It has been proved in [1] and [3] that \( E(\lambda_*, \lambda^*) \) is an uncountable set. In the present paper, we strengthen this result by showing that

\[
\dim E(\lambda_*, \lambda^*) \geq \frac{\lambda_* - \log \frac{1+\sqrt{5}}{2}}{2\lambda^*},
\]

where \( \dim \) denotes the Hausdorff dimension.

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1. Introduction

It is well known that every irrational number \( x \) can be expanded uniquely as an infinite continued fraction

\[
x = a_0(x) + \frac{1}{a_1(x) + \frac{1}{a_2(x) + \frac{1}{a_3(x) + \cdots}}},
\]

where \( a_0(x) \in \mathbb{Z} \) and \( a_i(x) \in \mathbb{N} \) for all \( i \geq 1 \). Without loss of generality, we always assume \( x \in [0, 1) \), i.e. \( a_0(x) \equiv 0 \).

For any \( n \geq 1 \) and \( \{a_1, a_2, \ldots, a_n\} \in \mathbb{N}^n \), we call

\[
I(a_1, a_2, \ldots, a_n) = \begin{cases} \frac{p_n}{q_n}, & \text{if } n \text{ is even,} \\ \frac{p_n+p_{n-1}}{q_n+q_{n-1}}, & \text{if } n \text{ is odd,} \end{cases}
\]
a CF-interval of rank $n$, where $p_k, q_k, 0 \leq k \leq n$, are defined by following recurrence relations.

\begin{align*}
p_{-1} &= 1; \quad p_0 = 0; \quad p_m = a_m p_{m-1} + p_{m-2}, \quad 1 \leq m \leq n. \quad (1) \\
q_{-1} &= 0; \quad q_0 = 1; \quad q_m = a_m q_{m-1} + q_{m-2}, \quad 1 \leq m \leq n. \quad (2)
\end{align*}

$I(a_1, a_2, \ldots, a_n)$ represents the set of numbers in $[0, 1)$ which have a continued fraction expansion begins by $a_1, a_2, \ldots, a_n$. It is well known, see [6], that

$$|I(a_1, a_2, \ldots, a_n)| = \frac{1}{q_n(q_n + q_{n-1})},$$

where $|I(a_1, a_2, \ldots, a_n)|$ denotes the length of $I(a_1, a_2, \ldots, a_n)$.

For an irrational number $x \in [0, 1)$, let

$$\beta_\ast(x) = \liminf_{n \to \infty} \frac{\log q_n(x)}{n},$$

$$\beta^\ast(x) = \limsup_{n \to \infty} \frac{\log q_n(x)}{n},$$

where $\{q_n(x) : n \geq 1\}$ is defined recursively by (2) using $\{a_n(x) : n \geq 1\}$. If $\beta_\ast(x) = \beta^\ast(x)$, we say that $x$ has a Lévy constant and denote the common value by $\beta(x)$. A famous result of P. Lévy [7] showed that for almost all $x$, we have

$$\beta(x) = \frac{\pi^2}{12 \log 2} \approx 1.18657.$$  

$\beta_\ast(x)$ and $\beta^\ast(x)$ describe the exponential growth rates of $q_n(x)$ in $n$. Faivre [2] showed that every quadratic number has a Lévy constant. It is easy to see that for any irrational number $x$, we have $\beta_\ast(x) \geq \log \frac{\sqrt{5} + 1}{2} := G$. Faivre [3] employed the ergodic theorem to prove that for any $G \leq \lambda < \infty$, there exists an $x \in [0, 1)$ such that $\beta(x) = \lambda$. Baxa [1] showed the following more general result by elementary means.

**Theorem 1.1** ([1]). For any $G \leq \lambda_\ast \leq \lambda^\ast < \infty$, there exist uncountably many $x \in [0, 1)$ such that

$$\beta_\ast(x) = \lambda_\ast \quad \text{and} \quad \beta^\ast(x) = \lambda^\ast.$$  

In this note, we improve Baxa’s result and show that

**Theorem 1.2.** For any $G \leq \lambda_\ast \leq \lambda^\ast < \infty$, let

$$E(\lambda_\ast, \lambda^\ast) = \{x \in [0, 1) : \beta_\ast(x) = \lambda_\ast, \beta^\ast(x) = \lambda^\ast\},$$

then

$$\dim E(\lambda_\ast, \lambda^\ast) \geq \frac{\lambda_\ast - G}{2\lambda^\ast},$$

where $\dim$ denotes the Hausdorff dimension.