Multi-index Filtrations and Generalized Poincaré Series

By
A. Campillo¹,‡, F. Delgado¹,‡, and S. M. Gusein-Zade²,‡

¹ University of Valladolid, Spain
² Moscow State University, Russia

Communicated by P. Michor
Received April 5, 2005; accepted in final form July 18, 2006
Published online January 19, 2007 © Springer-Verlag 2007

Abstract. A multi-index filtration on the ring of germs of functions can be described by its Poincaré series. We consider a finer invariant (or rather two invariants) of a multi-index filtration than the Poincaré series generalizing the last one. The construction is based on the fact that the Poincaré series can be written as a certain integral with respect to the Euler characteristic over the projectivization of the ring of functions. The generalization of the Poincaré series is defined as a similar integral with respect to the generalized Euler characteristic with values in the Grothendieck ring of varieties. For the filtration defined by orders of functions on the components of a plane curve singularity $C$ and for the so called divisorial filtration for a modification of $(C^2, 0)$ by a sequence of blowing-ups there are given formulae for this generalized Poincaré series in terms of an embedded resolution of the germ $C$ or in terms of the modification respectively. The generalized Euler characteristic of the extended semigroup corresponding to the divisorial filtration is computed giving a curious “motivic version” of an A’Campo type formula.

2000 Mathematics Subject Classification: 14H20, 32S99
Key words: Poincaré series, generalized Euler characteristic, multi-index filtrations, curve singularities

Introduction

In what follows we consider multi-index filtrations on the ring $\mathcal{O}_{V, 0}$ of functions on a germ $(V, 0)$ of a complex analytic variety, though some of the definitions and the constructions make sense in a more general setting. A one index filtration

$$\mathcal{O}_{V, 0} = J_0 \supset J_1 \supset \cdots \supset J_n \supset \cdots$$

on the ring (vector space) $\mathcal{O}_{V, 0}$ can be described by its Poincaré series

$$P(t) = \sum_{i=0}^{\infty} \dim(J_i/J_{i+1}) \cdot t^i$$

(when all the factors $J_i/J_{i+1}$ are finite dimensional). Such a filtration can be defined by the function $v(g) = \sup\{i : g \in J_i\}$ ($v : \mathcal{O}_{V, 0} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$) possessing the properties: $v(\lambda g) = v(g)$ for $\lambda \in \mathbb{C}^*$, $v(g_1 + g_2) \geq \min\{v(g_1), v(g_2)\}$. A multi-

- First two authors were partially supported by the grant MEC, PN I+D+i MTM2004-00958.
- Partially supported by the grants RFBR-04-01-00762, NSh-4719.2006.1 The author is thankful to the University of Valladolid for hospitality.
An example of a multi-index filtration considered in this paper is the following one. Let $(C, 0) \subset (\mathbb{C}^n, 0)$ be a germ of a reduced analytic curve, let $L = \bigcup_{k=1} C_k$ be its decomposition into irreducible components, and let $\varphi_k : (\mathbb{C}, 0) \to (\mathbb{C}^n, 0)$, $k = 1, \ldots, r$, be parametrizations of the components $C_k$ of the curve $C$; i.e., germs of analytic maps such that $\text{Im} \varphi_k = C_k$ and $\varphi_k$ is an isomorphism between $\mathbb{C}$ and $C_k$ outside of the origin (in a neighbourhood of it). For a germ $g \in \mathcal{O}_{\mathbb{C}^n, 0}$, let $v_k = v_k(g)$ and $a_k = a_k(g)$ be the power of the leading term and the coefficient at it in the power series decomposition of the germ $g \circ \varphi_k : (\mathbb{C}, 0) \to \mathbb{C}$, $g \circ \varphi_k(\tau) = a_k \tau^{v_k} + \text{terms of higher degree}$, $a_k \neq 0$. If $g \circ \varphi_k(\tau) \equiv 0$, $v_k(g)$ is assumed to be equal to $\infty$ and $a_k(g)$ is not defined. Let $v(g) := (v_1(g), \ldots, v_r(g)) \in \mathbb{Z}^r_{\geq 0}$, $a(g) := (a_1(g), \ldots, a_r(g)) \in (\mathbb{C}^*)^r$.

The functions (valuations) $v_k$ define a multi-index filtration on the ring $\mathcal{O}_{\mathbb{C}^n, 0}$: for $g \in \mathbb{Z}^r$, the corresponding subspace $J(v)$ is defined as $\{g \in \mathcal{O}_{\mathbb{C}^n, 0} : v(g) \geq v\}$. In [2] there was computed the (appropriately defined) Poincaré series of the described multi-index filtration on the ring $\mathcal{O}_{\mathbb{C}^2, 0}$ for a plane curve singularity $C$ (i.e., for $n = 2$). It appeared to be equal to the Alexander polynomial of the algebraic link $C \cap S^3_\varepsilon \subset S^3_\varepsilon$ corresponding to the curve $C$ ($S^3_\varepsilon$ is the sphere of radius $\varepsilon$ centred at the origin of $\mathbb{C}^2$ with positive $\varepsilon$ small enough).

Inspired by the notion of motivic integration (see, e.g., [8], [12]) there was defined the notion of the Euler characteristic of (some) subsets of the ring $\mathcal{O}_{V, 0}$ of functions on a germ $(V, 0)$ of an analytic variety or of its projectivization $\mathbb{P}\mathcal{O}_{V, 0}$ and the corresponding notion of the integration with respect to the Euler characteristic (see, e.g., [10]). Here the Euler characteristic can be considered both as the usual one $\chi$ with values in $\mathbb{Z}$ and the generalized one $\chi_g$ with values in the Grothendieck ring of complex algebraic varieties localized by the class $L$ of the complex affine line: $K_0(\mathbb{P}\mathcal{O}_{\mathbb{C}})(L)$. It was shown that the Poincaré series $P(t_1, \ldots, t_r)$ of a multi-index filtration $\{J(v)\}$ on the ring $\mathcal{O}_{V, 0}$ (finitely determined: see the definition below) is equal to the integral with respect to the Euler characteristic over the projectivization $\mathbb{P}\mathcal{O}_{V, 0}$ of the ring of functions:

$$P(t_1, \ldots, t_r) = \int_{\mathbb{P}\mathcal{O}_{V, 0}} t^{v(g)} d\chi$$

(1)

$(t = (t_1, \ldots, t_r), t^v = t_1^{v_1} \cdots t_r^{v_r})$. Moreover, this representation permitted to give a considerably shorter proof of the mentioned formula for the Poincaré series of the filtration defined by orders $v_i(g)$ of a function on the components of the plane curve singularity $C$ (see [3]) and to compute Poincaré series of some other multi-index filtrations; e.g. [7], [4].

Here we define a finer invariant (or rather two invariants) of a multi-index filtration than the Poincaré series $P(t_1, \ldots, t_r)$ which can be considered as generalizations of the last one. To define the first of them we consider an integral like the one in