Vector-valued Jacobi-like forms

By

Min Ho Lee

University of Northern Iowa, Cedar Falls, IA, USA

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Abstract. We introduce vector-valued Jacobi-like forms associated to a representation \( \rho : \Gamma \to GL(n, \mathbb{C}) \) of a discrete subgroup \( \Gamma \subset SL(2, \mathbb{C}) \) in \( \mathbb{C}^n \) and establish a correspondence between such vector-valued Jacobi-like forms and sequences of vector-valued modular forms of different weights with respect to \( \rho \). We determine a lifting of vector-valued modular forms to vector-valued Jacobi-like forms as well as a lifting of scalar-valued Jacobi-like forms to vector-valued Jacobi-like forms. We also construct Rankin-Cohen brackets for vector-valued modular forms.

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1. Introduction

Jacobi-like forms are power series, whose coefficients are holomorphic functions on the Poincaré upper half plane \( \mathcal{H} \), satisfying a certain transformation formula with respect to an action of a discrete subgroup \( \Gamma \) of \( SL(2, \mathbb{R}) \). This transformation formula is essentially one of the two conditions that must be satisfied by a Jacobi form (cf. [2]) and it determines relations among coefficients of the given Jacobi-like form. Such relations can be used to express each coefficient of a Jacobi-like form in terms of derivatives of some modular forms for \( \Gamma \). These modular forms belong to a sequence of the form \( \{ f_w \}_{w \geq 0} \), where \( f_w \) is a modular form of weight \( 2w + \lambda \) for some \( \lambda \in \mathbb{Z} \). In fact, sequences of modular forms for \( \Gamma \) of such type are in one-to-one correspondence with Jacobi-like forms for \( \Gamma \) of weight \( \lambda \). Another interesting aspect of Jacobi-like forms is that a Jacobi-like form for \( \Gamma \) determines a \( \Gamma \)-invariant pseudodifferential operator, which is a formal Laurent series in \( \partial^{-1} \) having holomorphic functions on \( \mathcal{H} \) as coefficients. Various topics related to mutual correspondences among Jacobi-like forms, pseudodifferential operators, and sequences of modular forms were investigated by Cohen, Manin, and Zagier (see [1] and [6]). Some of those results were extended to the case of several variables in [4].

Vector-valued modular forms for a discrete subgroup \( \Gamma \subset SL(2, \mathbb{R}) \) are \( \mathbb{C}^n \)-valued holomorphic function on \( \mathcal{H} \) for some positive integer \( n \) defined by an automorphy factor involving a representation \( \rho : \Gamma \to GL(n, \mathbb{C}) \) of \( \Gamma \) in \( \mathbb{C}^n \), and they play an important role in number theory. In [3] Kuga and Shimura studied
vector-valued modular forms associated to usual complex-valued modular forms. Connections between vector-valued modular forms and Jacobi-like forms were considered in [5], where vector-valued modular forms of weight two were constructed in terms of coefficients of Jacobi-like forms by using the method of Kuga and Shimura.

In this paper we introduce vector-valued Jacobi-like forms associated to \( \rho \) and extend some of the results of Cohen, Manin, and Zagier to the vector-valued case. In particular, we establish a correspondence between vector-valued Jacobi-like forms and sequences of vector-valued modular forms and determine a lifting of vector-valued modular forms to vector-valued Jacobi-like forms. As an application, we construct a lifting of scalar-valued Jacobi-like forms to vector-valued Jacobi-like forms as well as Rankin-Cohen brackets for vector-valued modular forms.

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2. Vector-valued Jacobi-like forms

In this section we introduce vector-valued Jacobi-like forms and establish a correspondence between such Jacobi-like forms and certain sequences of vector-valued modular forms.

Let \( \mathcal{H} \) be the Poincaré upper half plane, and let \( R \) be the space of holomorphic functions on \( \mathcal{H} \). We denote by \( R[[X]] \) the complex algebra of formal power series in \( X \) with coefficients in \( R \). We fix a positive integer \( n \) and denote by \( \hat{R} \) the space of \( \mathbb{C}^n \)-valued holomorphic functions on \( \mathcal{H} \). Then the space \( \hat{R}[[X]] \) of formal power series in \( X \) with coefficients in \( R \) has the structure of a module over \( R[[X]] \). Note that, unlike \( R[[X]] \), \( \hat{R}[[X]] \) does not have a ring structure. Let \( \Gamma \) be a discrete subgroup of \( SL(2, \mathbb{R}) \), and let \( \rho : \Gamma \to GL(n, \mathbb{C}) \) be a representation of \( \Gamma \) in \( \mathbb{C}^n \). Then the associated action of \( \Gamma \) on the coefficients induces an action of \( \Gamma \) on \( \hat{R}[[X]] \). Throughout the rest of this paper we fix \( \lambda \in \mathbb{Z} \) and \( \mu \in \mathbb{R} \).

Definition 2.1. (i) A Jacobi-like form of weight \( \lambda \) and index \( \mu \) for \( \Gamma \) is a formal power series \( \Phi(z, X) \in R[[X]] \) satisfying

\[
\Phi(\gamma z, (cz+d)^{-2}X) = (cz+d)^{\lambda} e^{\mu X/(cz+d)} \Phi(z, X)
\]

for all \( z \in \mathcal{H} \) and \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \), where \( \gamma z = (az+b)(cz+d)^{-1} \). We denote by \( \mathcal{J}_{\lambda, \mu}(\Gamma) \) the space of Jacobi-like forms of weight \( \lambda \) and index \( \mu \) for \( \Gamma \).

(ii) A vector-valued Jacobi-like form of weight \( \lambda \) and index \( \mu \) for \( \Gamma \) with respect to \( \rho \) is a formal power series \( \hat{\Phi}(z, X) \in \hat{R}[[X]] \) satisfying

\[
\hat{\Phi}(\gamma z, (cz+d)^{-2}X) = (cz+d)^{\lambda} e^{\mu X/(cz+d)} \rho(\gamma) \hat{\Phi}(z, X)
\] (2.1)

for all \( z \in \mathcal{H} \) and \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \). We denote by \( \mathcal{J}_{\lambda, \mu}(\Gamma, \rho) \) the space of vector-valued Jacobi-like forms of weight \( \lambda \) and index \( \mu \) for \( \Gamma \) with respect to \( \rho \).

If \( f \in \mathcal{R}, \hat{f} \in \hat{R}, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}) \) and \( \ell \in \mathbb{Z} \), then we set

\[
(f|_{\gamma})(z) = (cz+d)^{-\ell} f(\gamma z), \quad (\hat{f}|_{\gamma})(z) = (cz+d)^{-\ell} \hat{f}(\gamma z)
\]