Symmetric group actions on homotopy $S^2 \times S^2$

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Abstract. Let $X$ be a closed smooth 4-manifold which is homotopy equivalent to $S^2 \times S^2$. In this paper we use Seiberg-Witten theory to prove that if $X$ admits a spin symmetric group $S_4$ action of even type with $b_2^+(X) = b_2^-(X)$, then as an element of $R(S_4)$, $\text{Ind}_{S_4}D_X = k_1(1 - \theta) + k_2(\psi_1 - \psi_2)$ for some integers $k_1$ and $k_2$, where $1, \theta, \psi_1, \psi_2$ are irreducible characters of $S_4$ of degree 1, 1, 3, and 3 respectively.

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1. Introduction

Let $X$ be a smooth, closed, connected spin 4-manifold. We denote by $b_2(X)$ the second Betti number and denote by $\sigma(X)$ the signature of $X$. In [9], Matsumoto conjectured the following inequality

$$b_2(X) \geq \frac{11}{8}|\sigma(X)|. \quad (1)$$

This conjecture is well known and has been called the $\frac{11}{8}$-conjecture.

From the classification of unimodular even integral quadratic forms and Rochlin’s theorem, for the choice of orientation with non-positive signature the intersection form of a closed spin 4-manifold $X$ is

$$-2kE_8 \oplus mH, \quad k \geq 0,$$

where $E_8$ is the $8 \times 8$ intersection form matrix and $H$ is the hyperbolic matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus, $m = b_2^+(X)$ and $k = -\sigma(X)/16$ and so the inequality (1) is equivalent to $m \geq 3k$. Since $K3$ surface satisfies the equality with $k = 1$ and $m = 3$, the coefficient $\frac{11}{8}$ is optimal, if the $\frac{11}{8}$-conjecture is true.

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Donaldson has proved that if \( k > 0 \) then \( m \geq 3 \) [4]. In early 1995, using the Seiberg-Witten theory introduced by Seiberg and Witten [11], Furuta [6] proved that

\[
b_2(X) \geq \frac{5}{4} |\sigma(X)| + 2. \tag{2}
\]

This estimate has been dubbed the \( \frac{10}{3} \)-theorem. Inequality (2) follows by a surgery argument from the non-positive signature, \( b_1(X) = 0 \) case:

**Theorem 1** (Furuta [6]). Let \( X \) be a smooth spin 4-manifold with \( b_1(X) = 0 \) with non-positive signature. Let \( k = -\sigma(X)/16 \) and \( m = b_2^+(X) \). Then,

\[
2k + 1 \leq m
\]

if \( m \neq 0 \).

His key idea is to use a finite dimensional approximation of the monopole equation.

In [2], Bryan (see also [5]) used Furuta’s technique of finite dimensional approximation and the equivariant \( K \)-theory to improve the above bound by \( p \) under the assumption that \( X \) has a spin odd type \( \mathbb{Z}/2p \)-action satisfying some non-degeneracy conditions analogous to the condition \( m \neq 0 \). Later Kim [7] gave the same bound for smooth, spin, even type \( \mathbb{Z}/2p \)-action on \( X \) satisfying some non-degeneracy conditions analogous to Bryan’s.

In the paper [8], we used the same method to study the spin \( S_4 \) actions of even type on spin 4-manifold \( X \), we proved that if \( X \) admits a spin \( S_4 \) action of even type, then \( b_2^+(X) \geq |\sigma(X)|/8 + 3 \) under some non-degeneracy conditions.

In the present paper we would like to use Furuta’s technique of finite dimensional approximation and the equivariant \( K \)-theory to study the \( S_4 \) actions on homotopy \( S^2 \times S^2 \) of even type.

To state our main result, we need some preliminaries.

Let \( X \) be a smooth, closed and connected spin 4-manifold. Suppose that \( X \) admits a spin structure preserving action by a compact Lie group (or finite group) \( G \). We may assume a Riemannian metric on \( X \) so that \( G \) acts by isometries. This \( G \)-action can always be lifted to \( \hat{G} \)-actions on the spinor bundles, where \( \hat{G} \) is the following extension

\[
1 \rightarrow \mathbb{Z}/2 \rightarrow \hat{G} \rightarrow G \rightarrow 1.
\]

Recall that the \( G \)-action is of even type if \( \hat{G} \) contains a subgroup isomorphic to \( G \), and in turn is of odd type, otherwise.

For the spin \( G \)-action on \( X \) of even type, the Dirac operator \( D_X \) is \( G \)-equivariant and so, \( \text{Ind}_G D_X = \ker D_X - \text{coker} D_X \in R(G) \). In particular, if \( G = S_4 \), \( \text{Ind}_{S_4} D_X = a + b\theta + c\eta + d\psi_1 + e\psi_2 \in R(S_4) \), where \( 1, \theta, \eta, \psi_1 \) and \( \psi_2 \) are 5 irreducible characters of \( S_4 \) of degree 1, 1, 2, 3 and 3 respectively (for detail see Section 3), \( a, b, c, d \) and \( e \) are integers.

Our main result of this paper is as follows:

**Theorem 2.** Let \( X \) be a closed smooth 4-manifold which is homotopy equivalent to \( S^2 \times S^2 \). If \( X \) admits a spin \( S_4 \) action of even type with \( b_2^+(X/S_4) = b_2^+(X) \),