On power deformations of univalent functions

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Abstract For an analytic function $f(z)$ on the unit disk $|z| < 1$ with $f(0) = f'(0) - 1 = 0$ and $f(z) \neq 0$, $0 < |z| < 1$, we consider the power deformation $f_c(z) = z(f(z)/z)^c$ for a complex number $c$. We determine those values $c$ for which the operator $f \mapsto f_c$ maps a specified class of univalent functions into the class of univalent functions. A little surprisingly, we will see that the set is described by the variability region of the quantity $zf'(z)/f(z)$, $|z| < 1$, for most of the classes that we consider in the present paper. As an unexpected by-product, we show boundedness of strongly spirallike functions.

Keywords Univalent function · Variability region · Spirallike function

Mathematics Subject Classification (2000) Primary 30C45; Secondary 30C55

1 Introduction

Let $\mathcal{A}$ denote the set of analytic functions on the unit disk $\mathbb{D} = \{z : |z| < 1\}$ of the complex plane $\mathbb{C}$. Set furthermore $\mathcal{A}_0 = \{f \in \mathcal{A} : f(0) = 1\}$ and $\mathcal{A}_1 = \{f \in \mathcal{A} : f(0) = 0\}$.
We note that a function \( h(z) \) belongs to \( A_0 \) if and only if the function \( zh(z) \) belongs to \( A_1 \). In what follows, \( f(z)/z \) will be regarded as a function in \( A_0 \) for \( f \in A_1 \). More concretely, for a function \( f(z) = z + a_2z^2 + a_3z^3 + \cdots \) in \( A_1 \), the function \( f(z)/z \) is regarded as the analytic function \( 1 + a_2z + a_3z^2 + \cdots \).

Let \( A_0^\lambda \) be the set of invertible elements of \( A_0 \) with respect to ordinary multiplication; that is, \( A_0^\lambda = \{ h \in A_0 : h(z) \neq 0, \ z \in \mathbb{D} \} \). In what follows, \( \text{Log} \) means the (single-valued) analytic branch of \( \log h \) in \( \mathbb{D} \) determined by \( \text{Log} h(0) = 0 \) for \( h \in A_0^\lambda \).

We also set \( \text{Arg} h = \text{Im} \log h \) for \( h \in A_0^\lambda \). We note that \( \text{Log} \) maps \( A_0^\lambda \) bijectively onto the complex vector space \( V = \{ f \in A : f(0) = 0 \} \).

The set \( S \) consisting of all univalent functions in \( A_1 \) has been the central object to study in the theory of univalent functions since early twentieth century.

We are interested in classical subclasses of \( S \) in the present paper. Let us now introduce them. A function \( f \in A_1 \) is called convex if \( f \) maps \( \mathbb{D} \) univalently onto a convex domain in \( \mathbb{C} \). We denote by \( K \) the class of convex functions. It is well known that \( f \in A_1 \) is convex if and only if

\[
\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > 0, \quad z \in \mathbb{D}.
\]

Let \( \lambda \) be a number with \(-\pi/2 < \lambda < \pi/2\). For a point \( a \neq 0 \) in \( \mathbb{C} \), the \( \lambda \)-spiral segment \([0, a]_\lambda \) is defined to be the set \( \{0\} \cup \{ a \exp(-te^{i\lambda}) : 0 \leq t < +\infty \} \). A domain \( \Omega \) in \( \mathbb{C} \) is called \( \lambda \)-spirallike (about the origin) if \([0, a]_\lambda \subset \Omega \) for every \( a \in \Omega \). In particular, a \( \lambda \)-spirallike domain is called starlike. A function \( f \in A_1 \) is called \( \lambda \)-spirallike if \( f \) maps \( \mathbb{D} \) univalently onto a \( \lambda \)-spirallike domain. The class of \( \lambda \)-spirallike functions will be denoted by \( SP(\lambda) \). Set \( SP = \bigcup_{-\pi/2 < \lambda < \pi/2} SP(\lambda) \). The class of starlike functions \( SP(0) \) is also denoted by \( S^* \). It is known that \( f \in A_1 \) is \( \lambda \)-spirallike if and only if

\[
\Re \left( e^{-i\lambda} \frac{zf'(z)}{f(z)} \right) > 0, \quad 0 < |z| < 1.
\]

For a real number \( 0 \leq \alpha \leq 1 \), a function \( f \in A_1 \) is called starlike of order \( \alpha \) if \( \Re(zf'(z)/f(z)) \geq \alpha, \ z \in \mathbb{D} \). Let \( S^*(\alpha) \) denote the set of starlike functions of order \( \alpha \). Similarly, for \( 0 < \alpha < 1 \), a function \( f \in A_1 \) is called strongly starlike of order \( \alpha \) if \( |\text{Arg}(zf'(z)/f(z))| < \pi\alpha/2, \ z \in \mathbb{D} \), and the set of those functions will be denoted by \( SS(\alpha) \).

We can extend strong starlikeness to strong spirallikeness in an obvious way. Let \( \lambda \in (-\pi/2, \pi/2) \) and \( 0 < \alpha < 1 \). A function \( f \in A_1 \) is called strongly \( \lambda \)-spirallike of order \( \alpha \) if

\[
|\text{Arg} \frac{zf'(z)}{f(z)} - \lambda| < \frac{\pi\alpha}{2}, \quad z \in \mathbb{D}.
\]

We denote by \( SP(\lambda, \alpha) \) the set of these functions. When we do not specify \( \lambda \) and \( \alpha \), we simply call the functions strongly spirallike. This sort of classes were first introduced by Bucka and Ciooza [4].

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