Local wave-front sets of Banach and Fréchet types, and pseudo-differential operators

Sandro Coriasco · Karoline Johansson · Joachim Toft

Received: 18 September 2011 / Accepted: 1 February 2012 / Published online: 24 February 2012 © Springer-Verlag 2012

Abstract Let $\omega, \omega_0$ be appropriate weight functions and $\mathcal{B}$ be an invariant BF-space. We introduce the wave-front set $WF_{\mathcal{B}}(f)$ with respect to the weighted Fourier Banach space $\mathcal{B} = \mathcal{F}\mathcal{B}(\omega)$. We prove that the usual mapping properties for pseudo-differential operators $\text{Op}_t(a)$ with symbols $a$ in $S^{(\omega_0)}_{\rho,0}$ hold for such wave-front sets. In particular we prove $WF_{\mathcal{C}}(\text{Op}_t(a)f) \subseteq WF_{\mathcal{B}}(f)$ and $WF_{\mathcal{B}}(f) \subseteq WF_{\mathcal{C}}(\text{Op}_t(a)f) \cup \text{Char}(a)$. Here $\mathcal{C} = \mathcal{F}\mathcal{B}(\omega/\omega_0)$ and $\text{Char}(a)$ is the set of characteristic points of $a$.

Keywords Wave-front · Fourier · Banach · Modulation · Micro-local

Mathematics Subject Classification (2000) 35A18 · 35S30 · 42B05 · 35H10

1 Introduction

In this paper we consider (local) wave-front sets with respect to appropriate Banach and Fréchet spaces. Especially we focus on the case when these spaces agree with
Fourier images of translation invariant Banach function spaces (BF-spaces). The family of such wave-front sets contains the wave-front sets of Sobolev type, introduced by Hörmander [24], the classical wave-front sets (cf. Sect. 8.1 and 8.2 in [23]), and wave-front sets of Fourier Lebesgue types, introduced in [29]. Roughly speaking, for any given distribution $f$ and for appropriate Banach (or Fréchet) space $B$ of temperate distributions, the wave-front set $WF_B(f)$ of $f$ consists of all pairs $(x_0, \xi_0)$ in $\mathbb{R}^d \times (\mathbb{R}^d \setminus 0)$ such that no localizations of the distribution at $x_0$ belongs to $B$ in the direction $\xi_0$.

We also establish mapping properties for a quite general class of pseudo-differential operators on such wave-front sets, and show that the micro-local analysis in [29] in background of Fourier Lebesgue spaces can be further generalized. It follows that our approach gives rise to flexible micro-local analysis tools which fit well to the most common approach developed in e.g. [23,24]. In particular, we prove that usual mapping properties, which are valid for classical wave-front sets (cf. Chapters VIII and XVIII in [23]), also hold for wave-front sets of Fourier BF-types. For example, we show that for an appropriate space $C$ which is completely determined by $B$ and the symbol class for $a$ we have

$$WF_C(Op_t(a) f) \subseteq WF_B(f) \subseteq WF_C(Op_t(a) f) \cup \text{Char}(a).$$

(1)

That is, any operator $Op_t(a)$ shrinks the wave-front sets and opposite embeddings can be obtained by including $\text{Char}(a)$, the set of characteristic points of the operator symbol $a$.

The symbol classes for the pseudo-differential operators are given by $S^{(\omega_0)}_{\rho, \delta}(\mathbb{R}^{2d})$, the set of all smooth functions $a$ on $\mathbb{R}^{2d}$ such that $a/\omega_0 \in S^0_{\rho, \delta}(\mathbb{R}^{2d})$. Here $\rho, \delta \in \mathbb{R}$ and $\omega_0$ is an appropriate smooth function on $\mathbb{R}^{2d}$. We note that $S^{(\omega_0)}_{\rho, \delta}(\mathbb{R}^{2d})$ agrees with the Hörmander class $S^r_{\rho, \delta}(\mathbb{R}^{2d})$ when $\omega_0(x, \xi) = \langle \xi \rangle^r$, where $r \in \mathbb{R}$ and $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$.

The set of characteristic points $\text{Char}(a)$ of $a \in S^{(\omega_0)}_{\rho, \delta}$ is the same as in [29], and depends on the choices of $\rho, \delta$ and $\omega_0$ (see Definition 2 and Proposition 5). We recall that this set is smaller than the set of characteristic points given by [23]. It is empty when $a$ satisfies a local ellipticity condition with respect to $\omega_0$, which is fulfilled for any hypoelliptic partial differential operator with constant coefficients (cf. [29]). As a consequence of (1), it follows that such hypoelliptic operators preserve the wave-front sets, as expected (cf. Example 4.9 in [29]).

In view of their definition, the information about regularity of distributions in background of the wave-front sets of Fourier BF-types might be more detailed compared to classical wave-front sets, because the family of Fourier BF-spaces is broad and such spaces can locally be chosen to be “arbitrary close” to $C^\infty$, the set of smooth functions. In this context, the classical wave-front set is exactly the wave-front set with respect to $C^\infty$. For example, the space $\mathcal{FB}(\omega) = \mathcal{F}L^1_{(\omega)}(\mathbb{R}^d)$, with $\omega(\xi) = \langle \xi \rangle^N$ for some integer $N \geq 0$, is locally close to $C^N(\mathbb{R}^d)$ (cf. the introduction of [29]). Consequently,