Spectral Properties of Interval Exchange Maps

By

Juliana Chaves\textsuperscript{1} and Arnaldo Nogueira\textsuperscript{1,2,*}

\textsuperscript{1}Universidade Federal do Rio de Janeiro, Brazil
\textsuperscript{2}Institut de Mathématiques de Luminy, Marseille, France

(Received 1 February 2001)

Abstract. In [7], Nogueira and Rudolph proved that for irreducible permutations not of rotation class almost every (a.e.) interval exchange transformation (i.e.t.) is topological weak mixing. It is conjectured that the claim holds if topological weak mixing is replaced by weak mixing. Here we study the behaviour of eigenfunctions of i.e.t. Our analysis gives alternative proofs of results due to Katok and Stepin [4] and Veech [10]: for certain permutations a.e. i.e.t. is weak mixing and for irreducible permutations a.e. i.e.t. is totally ergodic.

2000 Mathematics Subject Classification: 37A05, 37A30
Key words: Interval exchange transformations, weakly mixing

1. Introduction

This article is concerned with spectral properties of interval exchange transformations (i.e.t.). Let $m \in \mathbb{Z}$, $m > 1$, be fixed. In order to define the notions we need, it is necessary to recall some notations:

$$\Lambda_m = \{ \lambda \in \mathbb{R}^m : \lambda_i > 0, 1 \leq i \leq m \}.$$

For $\lambda \in \Lambda_m$, we set

$$\beta_0(\lambda) = 0, \beta_i(\lambda) = \beta_{i-1}(\lambda) + \lambda_i, \ 1 \leq i \leq m,$$

$$|\lambda| = \lambda_1 + \cdots + \lambda_m,$$

$$I^\lambda = [0, |\lambda|), \quad I_i^\lambda = [\beta_{i-1}, \beta_i), \ 1 \leq i \leq m.$$

We denote by $S_m$ the group of permutations on $\{1, \ldots, m\}$ and, for $\pi \in S_m$, $\lambda \in \Lambda_m$, we set $\lambda_\pi = \lambda_{\pi^{-1}}$.

The $(\lambda, \pi)$-interval exchange is the one-one onto map, $T = T_{(\lambda, \pi)}$, of $I^\lambda$ defined by, for $x \in I_i^\lambda$, $1 \leq i \leq m$,

$$Tx = x - \beta_{i-1}(\lambda) + \beta_{\pi i-1}(\lambda^\pi).$$

$T$ is irreducible, if $T[0, y) = [0, y)$, for $y > 0$, implies $y = |\lambda|$. It is equivalent to say $\pi$ is irreducible, which means that $\pi\{1, \ldots, k\} = \{1, \ldots, k\}$ implies that $k = m$. We denote by $S_m^0$ the set of irreducible permutations.

*Partially supported by grants from CNPq-Brazil, 301456/80, and FINEP/CNPq/MCT 41.96.0923.00 (PRONEX).
The discontinuities of $T$ belong to $\{\beta_1, \ldots, \beta_{m-1}\}$. The map $T^{-1}$ is the interval exchange defined by $(\lambda^i, \pi^{-1})$ whose discontinuities belong to $\{T\beta_i : i = 0, 1, \ldots, m-1, i \neq \pi^{-1}i - 1\}$.

The paper is organized as follows. The Rauzy induction [8] is defined in Section 2. We study algebraic properties of permutations in Section 3. In Section 4 we state some conditions we need thereafter. In Section 5, we establish conditions that eigenfunctions fulfill for a.e. interval exchange. In the next sections our results are applied. In Section 6 we give an alternative proof that for certain permutations a.e. interval exchange is weakly mixing, theorems due to Katok and Stepin [4] and Veech [10]. Our approach reveals the algebraic structure of the permutations that forces the eigenvalue to be 1. We also show what happens to other permutations. In Section 7 we give a short proof that for irreducible permutations a.e. interval exchange is totally ergodic, a result proved in [10].

2. Rauzy Induction

In [8], Rauzy introduced an induction procedure for i.e.t. It has been used to prove most of the dynamical properties of the transformations. For more information on the subject we suggest Veech [9] and Nogueira and Rudolph [7]. Next we state the result needed to carry on Rauzy induction.

**Definition 2.1.** We say $\lambda$ is **irrational**, if $\lambda_1, \ldots, \lambda_m$ are linearly independent over $\mathbb{Q}$.

**Theorem 2.2** (Keane [5]). Let $\pi \in S_m^0$ and $\lambda \in \Lambda_m$ be irrational. Then two discontinuities of $T$ never lie on the same orbit and, moreover, every orbit of $T$ is dense on $I^\lambda$.

Let $T = T_{(\lambda, \pi)}$ satisfy the hypothesis of Theorem 2.2. If $\lambda_{\pi^{-1}m} < \lambda_m$, we denote by $U_1$, the Poincaré first return map induced by $T$ on the subinterval $I' = [0, |\lambda| - \lambda_{\pi^{-1}m})$. We have, for $x \in I'$,

$$U_1x = \begin{cases} T^2x, & \text{if } x \in I_{\pi^{-1}m} \\ Tx, & \text{otherwise.} \end{cases}$$

There exists $(\alpha, \tau) \in \Lambda_m \times S_m^0$ such that $U_1 = T_{(\alpha, \tau)}$. We notice that there exists a unique $m \times m$ matrix $E$ whose entries are 0 and 1 such that $\lambda = E\alpha$. Otherwise $\lambda_m < \lambda_{\pi^{-1}m}$, since $\lambda$ is irrational, and we call $U_2$, the Poincaré first return map induced by $T$ on the subinterval $I' = [0, |\lambda| - \lambda_m)$. We have, for $x \in I'$,

$$U_2x = \begin{cases} T^2x, & \text{if } x \in T^{-1}I_m \\ Tx, & \text{otherwise.} \end{cases}$$

There exists $(\alpha, \tau) \in \Lambda_m \times S_m^0$ such that $U_2 = T_{(\alpha, \tau)}$. We notice that there exists a unique $m \times m$ matrix $E$ whose entries are 0 and 1 such that $\lambda = E\alpha$. We set

$$T_1 = \begin{cases} U_1, & \text{if } \lambda_{\pi^{-1}m} < \lambda_m \\ U_2, & \text{otherwise.} \end{cases}$$

In both cases there exist $(\lambda', \pi_1) \in \Lambda_m \times S_m^0$ and a unique matrix $E_1$ such that $T_1 = T_{(\lambda', \pi_1)}$ and $\lambda = E_1\lambda'$. Since $\lambda$ is irrational the procedure can be carried on for