Ergodic Group Rotations, Hartman Sets and Kronecker Sequences

By

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Dedicated to Edmund Hlawka on the occasion of his 85th birthday

Abstract. Let \( \iota : \mathbb{Z} \to C \) be a homomorphism with dense image \( \iota(\mathbb{Z}) \) in the compact group \( C \). If \( M \subseteq C \) is a continuity set, i.e. its topological boundary has Haar measure 0, then \( S = \iota^{-1}(M) \) is called a Hartman set. If \( M \) is aperiodic then \( S \) contains the essential information about \( (C, \iota) \) or, equivalently, about the dynamical system \( (C, T) \) where \( T \) is the ergodic group rotation \( T : c \mapsto c + \iota(1) \). Using Pontryagin’s duality the paper presents a new method to get this information from \( S \). The set \( S \) induces a filter \( \mathcal{F} \) on \( \mathbb{Z} \) which is an isomorphism invariant for \( (C, T) \) and turns out to be a complete invariant for ergodic group rotations. If one takes \( C = \mathbb{T}^s, \iota : k \mapsto k\alpha, \alpha = (\alpha_1, \ldots, \alpha_s), s \in \mathbb{N}, \) one gets the interesting special case of Kronecker sequences \( \{n\alpha\} \) which are classical objects in number theory and diophantine analysis.

Key words: Monothetic groups, group rotations, group compactifications, Hartman sets, Hartman sequences, Kronecker sequences

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1. Introduction

Let \( C \) be a compact (including Hausdorff’s separation axiom) monothetic group (additively written) with Haar measure \( \mu_C \). Monothetic means that there is a topologically generating element \( g \in C \), i.e. whose multiples \( kg, k \in \mathbb{Z} \), are dense in \( C \). For given \( M \subseteq C \) let \( S = \{ k \in \mathbb{Z} : kg \in M \} \). What do we know about \( C \) and \( g \) if only the set \( S \) is given?

The classical approach (for more details see for instance [2], sections 9 and 15) connecting topological and symbolic dynamical systems is to consider a continuity set \( M \subseteq C \) which separates the points of \( C \) under the ergodic transformation \( T : x \mapsto x + g \). (A continuity set \( M \), by definition, has a topological boundary \( \delta M \) with \( \mu_C(\delta M) = 0 \).) For each \( c \in C \) one defines a two sided 0-1-sequence by \( a_k(c) = 1 \) for \( T^k(c) \in M \) and \( a_k(c) = 0 \) otherwise, \( k \in \mathbb{Z} \). In [4] and [12] such sequences have been called Hartman sequences, due to Hartman’s work on group compactifications and the distributions of sequences (cf. [5], furthermore [8] and [10]). Similarly we call \( S \subseteq \mathbb{Z} \) a Hartman set if it is, for some Hartman sequence, the set of all \( k \) with \( a_k(c) = 1 \). Let \( X = 2^\mathbb{Z} \) denote the (compact metric) space of all two sided 0-1-sequences, \( \sigma : (a_k)_{k \in \mathbb{Z}} \mapsto (a_{k+1})_{k \in \mathbb{Z}} \) the shift on \( X \), and \( Y \subseteq X \) the closure of the set of all \( (a_k(c))_{k \in \mathbb{Z}} \), \( c \in C \). Then the mapping \( (a_k(c))_{k \in \mathbb{Z}} \mapsto c \) has a
continuous extension \( \varphi \) to \( Y \) which is one-to-one on a set of full measure and satisfies \( T \varphi = \varphi \sigma \). It follows that \((Y, \sigma)\) and \((C, T)\) have the same discrete spectrum which is the subgroup \( A \) of \( \mathbb{T} = \mathbb{R}/\mathbb{Z} \) which consists of all \( \chi(g) \), where \( \chi \) runs through the dual of \( C \), i.e. the set of all characters (cf. [13], Theorem 3.5). \( A \) is a complete invariant for ergodic group rotations, i.e. determines \((C, T)\) up to conjugation (cf. [13], Chapter 3, in particular Theorem 3.4).

This paper, continuing the investigations in [4] and [12], presents a new method to obtain, for given \( S, A \) and hence the relevant information about \( C \) and \( g \). The main idea is the following. Given \( S = \{ k : k g = T^k(0) \in M \} \), one can define a filter \( \mathcal{F}(S) \) on \( \mathbb{Z} \) by using that all symmetric difference sets \( S_k = S \triangle (S + k) \), \( k \in \mathbb{Z} \), have an asymptotic density which is small if \( kg \) is sufficiently close to \( 0 \in C \). If \( M \) is aperiodic in the sense that all its proper translates \( M + c, c \neq 0 \), differ from \( M \) by a set of positive measure, then the filter \( \mathcal{F}(S) \) turns out to be the filter \( \mathcal{F}(C, \iota) \) generated by the preimages \( \iota^{-1}(U) \) of neighbourhoods \( U \) of \( 0 \in C \) under \( \iota : \mathbb{Z} \to C, \ k \mapsto kg \). \( \mathcal{F}(C, \iota) \) is an invariant of the dynamical system which uniquely determines \( A \) and hence \((C, T)\) respectively the (group) compactification \((C, \iota)\) of \( C \) up to equivalence (by definition \((C, \iota)\) is called a compactification of \( C \) if \( C \) is a compact group and \( \iota : \mathbb{Z} \to C \) is a homomorphism such that \( \iota(\mathbb{Z}) \) is dense in \( C \)).

Note that each compactification is a factor of the maximal Bohr compactification of \( C \). By Pontryagin’s duality factor groups correspond to subgroups \( A \) of the dual, which in this case is the discretely topologized one-dimensional torus 
\[ \mathbb{T}_d = (\mathbb{R}/\mathbb{Z})^d. \]
Thus the canonical isomorphisms between the systems \( \mathcal{C}(M \mathbb{P}) \) of all compactifications \((C, \iota)\) of \( \mathbb{Z} \), \( \mathcal{F} \otimes \mathcal{T} \) of all corresponding filters \( \mathcal{F}(C, \iota) \) and \( \mathcal{P}(\mathcal{M} \mathbb{P}) \) of all subgroups \( A \) of \( \mathbb{T}_d \) are fundamental in our construction.

To obtain the classical Kronecker sequences one has to consider \( C = \mathbb{T}^s = \mathbb{R}^s/\mathbb{Z}^s, s \in \mathbb{N} \), and a topologically generating element \( g = (\alpha_1, \ldots, \alpha_s) \), i.e. where the \( \alpha_i \) together with \( 1 \) are linearly independent over \( \mathbb{Q} \). The resulting subgroup \( A \) is generated by the \( \alpha_i \). Note that we cannot expect to recover the generators \( \alpha_i \) of \( A \) themselves since the generating system \( \{\alpha_1, \ldots, \alpha_s\} \) is not invariant under automorphisms of \( A \) (in the case \( s = 1 \) the group \( A \) is cyclic with only two generating elements, namely \( \alpha \) and \( -\alpha \), hence \( \alpha \) is unique up to its sign). Sander treats in [11] our topic in this context by using Kronecker’s approximation theorem, illustrating the connections to (simultaneous) diophantine approximation. For the quantitative theory consider for instance [1] and [9] to get an impression of recent developments. As mentioned in [1], a main problem in the quantitative theory is that for \( s > 1 \) there is no simple analogue to the theory of continued fraction expansions which is a very satisfactory and successful tool for \( s = 1 \). Concerning qualitative results, the filter approach in this paper might be a contribution to this problem.

The paper is organized as follows. After the introduction, Chapter 2 consists of three sections with several preliminaries. Section 2.1 introduces further notations and recalls several facts about group compactifications. Most of them are implications of Pontryagin’s duality theory applied to the locally compact group \( \mathbb{Z} \). Theorem 1 in Section 2.2 presents the announced isomorphisms between the (partially ordered) sets \( \mathcal{C}(M \mathbb{P}), \mathcal{T} \otimes \mathcal{F} \) and \( \mathcal{P}(\mathcal{M} \mathbb{P}) \). Section 2.3 introduces further notations concerning factorization modulo 0-sets and the nonnegative