Summatory Formula of the Convolution of Two Arithmetical Functions

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Abstract. We study the asymptotic formula of \( \sum_{n \leq x} f \ast g(n) \) for some arithmetical functions \( f \) and \( g \). This generalizes the case \( 1 \ast v(n) \) investigated by Balakrishnan and Pétermann.

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1. Introduction

Let \( u \) be an arithmetical function. The behaviour of its summatory function \( \sum_{n \leq x} u(n) \) is closely related to its associated Dirichlet series \( \mathcal{U}(s) = \sum_{n=1}^{\infty} u(n)n^{-s} \). Suppose \( \mathcal{U}(s) \) is meromorphic in the complex plane \( \mathbb{C} \) and satisfies a mild growth condition. Then Perron’s formula gives

\[
\sum_{n \leq x} u(n) = M(x) + E(x)
\]

where the dominating part \( M(x) \), called the main term, is given by the poles of \( \mathcal{U}(s) \), and \( E(x) \), called the error term, accounts for the deviation.

In [1], Balakrishnan and Pétermann considered the case \( u = 1 \ast v \) and so \( \mathcal{U}(s) = \zeta(s)Z(s) \) where \( \zeta(s) \) is the Riemann zeta-function and \( Z(s) = \sum_{n=1}^{\infty} v(n)n^{-s} \). They investigated the situation that \( Z(s) \) is absolutely convergent for \( \Re s > a \) and the analytic continuation of \( Z(s) \) behaves essentially like \( z^{\alpha}(s) \) where \( \alpha \in \mathbb{C} \) and \( z(s) \) is a meromorphic function which has a simple pole at \( s = a \). Their main result is the following asymptotic formula:

\[
\sum_{n \leq x} u(n) = Lx + x^a \sum_{m=0}^{R} b_m (\log x)^{\alpha-m-\rho} - \sum_{n \leq x/N(x)} v(n) \psi \left( \frac{x}{n} \right) + O(x^a N(x)^{-c'})
\]

(1.1)

where \( L, b_m, \alpha \) and \( c' \ (> 0) \) are constants, \( \rho = 0 \) or \( 1 \), \( \psi(x) = x - \lfloor x \rfloor - 1/2 \), \( R = R(x) \) and \( N(x) \) are positive increasing functions. This formula holds for \( 0 \leq a < 1 \), their proof (in [1, Section 3]) is slightly complicated and bulky. The easier case \( a < 0 \) is handled in [1, Section 2].
Modifying the approach, we can give a simpler proof for the case $0 \leq a < 1$, and furthermore, consider the general sum $\sum_{n \leq x} f(n)g(n)$ under certain assumptions. As well, our result can show the link between the singularities of $\mathcal{H}$ and the main term in a more descriptive form. It seems comparatively non-trivial to see this linkage from the proof in [1]. Finally, we shall give an application which is a generalization of a problem of Rényi.

Throughout this paper, $c$, $c'$ and $c_i$ denote some unspecified positive constants, and $\epsilon > 0$ denotes an arbitrarily small constant. Besides, we write $\log_2 x$ for $\log \log x$ and $\mathcal{H}_X(a, r)$ for the truncated Hankel contour starting and ending at $a - X$, and surrounding $s = a$ with radius $r$ (i.e. it is a positively oriented contour consisting of two straight line segments $[(a - X)e^{\pm i\pi}, (a - r)e^{\pm i\pi}]$ and the circular path $\{a + re^{i\theta} : -\pi < \theta < \pi\}$). Finally we use $U(a, r, X)$ to denote an open connected set which contains the line segment $[a - X, a]$ and the closed disc of radius $r$ centered at $a$; and $U^-(a, r, X) = U(a, r, X) \setminus (-\infty, a]$.

2. Assumptions and the Main Result

Given two complex-valued arithmetical functions $f$ and $g$, and three fixed numbers $0 \leq a < \eta < 1$ and $0 \neq \alpha < a$, we write $\mathcal{F}(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$, $\mathcal{G}(s) = \sum_{n=1}^{\infty} g(n)n^{-s}$ and assume the following conditions:

(1) $\mathcal{F}(s)$ and $\mathcal{G}(s)$ are absolutely convergent on $\Re s > 1$ and $\Re s > a$ respectively.

(2) $\mathcal{F}(s)$ has a meromorphic continuation on $\Re s > \alpha - \epsilon$ such that it has a finite number of poles in the region $\eta < \Re s \leq 1$ but no poles on the strip $\alpha \leq \Re s \leq \eta$. Moreover, we suppose for $\alpha \leq \sigma \leq 1$, and for all sufficiently large $|t|$, $\mathcal{F}(\sigma + it) \ll |t|^{1-\epsilon}$. \hspace{1cm} (2.1)

(3) $\mathcal{G}(s)$ can be analytically continued to a larger region such that it is holomorphic on $U^- = U^-(a, \delta, Y)$ and $\mathcal{G}(s) \ll r^{-c}$ if $|s - a| = r$ and $s \in U^-$, where $\delta < Y$ satisfies $0 < \delta < \eta - a$ and $Y < a - \alpha$.

(4) As $x \to \infty$, we have $G(x) = \sum_{n \leq x} g(n) = \Phi(x) + O(x^{\alpha}\exp(-c_0\mathcal{N}(x)))$ \hspace{1cm} (2.2)

where $\Phi(x) = \frac{1}{2\pi i} \int_{\mathcal{H}_X(a, \delta)} \mathcal{G}(s)x^s \frac{ds}{s}$.

The function $\mathcal{N}(x)$ is positive, non-decreasing and satisfying (i) $\mathcal{N}(x)/\log_2 x \to \infty$ and (ii) $\mathcal{N}(x^{\gamma}) \gg_{\gamma} \mathcal{N}(x)$ for some $\gamma \in (0, 1)$.

Remarks. The condition $\alpha \neq 0$ is not essential, it helps simplifying a little the proof. The conditions in (3) and (4) are not more restrictive than conditions (i)–(v) in [1, Section 3.1]. The formula (2.2) is often obtained by the Selberg-Delange