On the $L_p$-Discrepancy of the Hammersley Point Set

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Abstract. We give a formula for the $L_p$-discrepancy of the 2-dimensional Hammersley point set in base 2 for all integers $p$, $1 \leq p < \infty$.

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1. Introduction

For a point set $x_0, \ldots, x_{N-1}$ of points in the 2-dimensional unit-cube $[0,1)^2$ the discrepancy function is defined as

$$\Delta(\alpha, \beta) := A_N([0,\alpha) \times [0,\beta)) - N\alpha\beta,$$

where $0 < \alpha, \beta \leq 1$ and $A_N([0,\alpha) \times [0,\beta))$ denotes the number of $i$ with $x_i \in [0,\alpha) \times [0,\beta)$. Now the $L_p$-discrepancy, $p > 0$, of the point set is defined as the $L_p$-norm of the discrepancy function devided by the cardinality of the point set and is a measure for the irregularity of distribution of the point set over $[0,1)^2$ (see for example [3]), i.e., for $0 < p < \infty$ we set

$$L_p(x_0, \ldots, x_{N-1}) := \frac{1}{N} \left( \int_0^1 \int_0^1 |\Delta(\alpha, \beta)|^p \, d\alpha \, d\beta \right)^{1/p}.$$

For $p = \infty$ we get the usual star-discrepancy of the point set

$$L_\infty(x_0, \ldots, x_{N-1}) := \frac{1}{N} \sup_{0 < \alpha, \beta \leq 1} |\Delta(\alpha, \beta)|.$$

In this paper we consider the $L_p$-discrepancy of a very special point set in $[0,1)^2$, the well known Hammersley point set in base 2 with $N = 2^s$ points, which is defined as follows:

For $n = 0, 1, \ldots, 2^s - 1$ let

$$n = n_0 + n_1 2 + \cdots + n_{s-1} 2^{s-1}$$

be the base 2 representation of $n$. Define

$$x_n = \frac{n}{2^s} \text{ and } y_n = \frac{n_0}{2} + \frac{n_1}{2^2} + \cdots + \frac{n_{s-1}}{2^s}.$$
and set $x_n := (x_n, y_n)$. Then the point set $x_0, \ldots, x_{N-1}$ is the Hammersley point set in base 2 with $N = 2^s$ points in $[0, 1)^2$. In the following we shall denote this point set by $\mathcal{H}_s$.

In [2], [4] and [6], it was shown that for the star-discrepancy of $\mathcal{H}_s$ we have (with $N = 2^s$)

$$NL_\infty(\mathcal{H}_s) = \frac{s}{3} + \frac{13}{9} - \left(-1\right)^s \frac{4}{9 \cdot 2^s}$$  

(1)

and therefore

$$\lim_{N \to \infty} \frac{NL_\infty(\mathcal{H}_s)}{\log N} = \frac{1}{3 \log 2}.$$  

Further, Halton and Zaremba [4] proved the following result for the $L_2$-discrepancy of the Hammersley point set with $N = 2^s$ points. They showed

$$NL_2(\mathcal{H}_s) = \left(\frac{s^2}{64} + \frac{29s}{192} + \frac{3}{8} - \frac{s}{16 \cdot 2^s} + \frac{1}{4 \cdot 2^s} - \frac{1}{72 \cdot 2^{2s}}\right)^{1/2}$$  

(2)

and therefore

$$\lim_{N \to \infty} \frac{NL_2(\mathcal{H}_s)}{\log N} = \frac{1}{8 \log 2}.$$  

Now from (1) and (2) one sees that for all $p \geq 2$ the $L_p$-discrepancy of the Hammersley point set $\mathcal{H}_s$ (with $N = 2^s$ points) is of order $\frac{\log N}{N}$ (since the $L_p$-norm of a function is increasing in $p$).

Note that a simple modification of our point set results in an essentially smaller order of $L_2$-discrepancy. Consider the point set with $N = 2^{s+1}$ points

$$(x_n, y_n), (x_n, 1 - y_n); \ n = 0, 1, \ldots, 2^s - 1,$$

where $x_n$ and $y_n$ are defined as in the definition of the Hammersley point set $\mathcal{H}_s$. Then in [5] it was shown that this point set has $L_2$-discrepancy

$$L_2 \leq c \frac{\left(\log N\right)^{1/2}}{N},$$

with an absolute constant $c$ not depending on $N$. (This result is also valid in a more general setting; see [5].) Moreover, from Roth [8] we know that this order is best possible.

It is the aim of this paper to give a formula for the $L_p$-discrepancy of the Hammersley point set $\mathcal{H}_s$ for all integers $p$, $1 \leq p < \infty$, which gives the order of the $L_p$-discrepancy and the exact value of the constant at the “leading term” (Theorem 1). Moreover we calculate the value of

$$\lim_{N \to \infty} \frac{NL_p(\mathcal{H}_s)}{\log N}$$

(Corollary 1). Further we prove an exact formula for the $L_1$-discrepancy and once more for the $L_2$-discrepancy of the Hammersley point set $\mathcal{H}_s$ (Theorem 2).