Hierarchical Quadrature for Singular Integrals

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Abstract

We introduce a method for the computation of singular integrals arising in the discretization of integral equations. The basic method is based on the concept of admissible subdomains, known, e.g., from panel clustering techniques and \( \mathcal{H} \)-matrices: We split the domain of integration into a hierarchy of subdomains and perform standard quadrature on those subdomains that are amenable to it. By using additional properties of the integrand, we can significantly reduce the algorithmic complexity of our approach. The method works also well for hypersingular integrals.

AMS Subject Classifications: 65D32, 42B20.

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1. Introduction

We consider integrals as they arise in the boundary element method. Let us consider

\[
I := \int_0^1 \int_0^1 |x-y|^a \, dy \, dx
\]  

(1.1)

as an example. The kernel function is improperly integrable if \( a > 1 \). However, since also strongly singular integrals appear in the BEM, \( a \leq -1 \) will not be excluded. In the latter case, \( I \) has to be interpreted as partie finie integral in the sense of Hadamard [8, pp. 184].

The standard quadrature methods apply one of the follow techniques:

1. Use a quadrature rule adapted to the singularity of the kernel (cf. [17], [16]).
2. Apply an exact integration at least for one of the double integrals. Hopefully the remaining integral is regular and standard Gauss quadrature works well (cf. [12], [17]).
3. Apply a transformation that removes the singularity (cf. [2], [10], [11]).
4. Apply a transformation like \( \xi = x - y, \eta = x + y \), that changes the moving singularity \( |x-y|^a \) of (1.1) into the fixed singularity \( |\xi|^a \) and apply 1 or 2 (cf. [10]).
(5) Adaptive refinement, i.e., the integration region is split into suitable subregions. Subregions that do not contain the singularity are treated by standard quadrature, the remaining subregions are split again (cf. [18]).

Although the techniques 1–4 are quite successful in special situations, they are of limited use. Method 5 might be costly because of many levels of refinement. The mentioned techniques depend strongly on the dimensionality of the integral. In particular, all methods are not easily extended to strongly singular integrals.

In this paper, we describe a technique which combines the adaptive refinement from 5 with further structural properties of the integrand. Eventually, we have to determine only few integrals over smooth integrands and to solve a small system of equations. Moreover, this approach extends to strong singularities.

Our assumptions are explained for

\[ I := \int_0^1 \int_0^1 \kappa(x,y) dy \ dx. \]

We suppose that the integrand \( \kappa(x,y) \) – possibly after subtracting a smooth part – satisfies two conditions: translation invariance and homogeneity. The first condition reads

\[ \kappa(x,y) = \kappa(x+c,y+c) \text{ for all } c \in \mathbb{R}, \quad (1.2) \]

while a homogeneous integrand satisfies

\[ \kappa(sx, sy) = s^\alpha \kappa(x,y) \text{ for all } s \in \mathbb{R}_{>0}, \quad (1.3) \]

where \( \alpha \in \mathbb{R} \) is the degree of homogeneity. A closely related variant of (1.3) occurs, e.g., for \( \kappa(x,y) = \log(|x-y|) \), where

\[ \kappa(sx, sy) = \beta(s) + \kappa(x,y) \text{ for all } s \in \mathbb{R}_{>0}. \quad (1.4) \]

Furthermore, \( \kappa \) is assumed to be sufficiently smooth outside of a neighborhood of the possible singularity at \( x = y \) (A possible qualification of the smoothness of \( \kappa \) is given by the asymptotic smoothness, cf. (2.13)).

In addition to (1.2), a symmetry condition

\[ \kappa(y,x) = \kappa(x,y) \quad (1.5) \]

can be exploited (antisymmetry \( \kappa(y,x) = -\kappa(x,y) \) would lead immediately to the trivial result \( I = 0 \)).