Signal and image approximation with level-set constraints

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Received 27 January 2007; Accepted 21 August 2007; Published online 6 November 2007
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Summary

We present a novel variational approach to signal and image approximation using filter statistics (histograms) as constraints. Given a set of linear filters, we study the problem to determine the closest point to given data while constraining the level-sets of the filter outputs. This criterion and the constraints are formulated as a bilevel optimization problem. We develop an algorithm by representing the lower-level problem through complementarity constraints and by applying an interior-penalty relaxation method. Based on a decomposition of the penalty term into the difference of two convex functions, the resulting algorithm approximates the data by solving a sequence of convex programs. Our approach allows to model and to study the generation of image structure through the interaction of two convex processes for spatial approximation and for preserving filter statistics, respectively.

AMS Subject Classifications: 68U10; 65K05; 65K10; 90C33.

Keywords: level-sets; image approximation; equilibrium constraints; complementarity constraints; DC-programming.

1. Introduction

Filter statistics play an important role in both natural and computational vision systems [8], [12], [20]. It has been shown, for instance, that the statistics (histograms) of bandpass filter outputs, collected over a large image database, are significant for natural scenes and can be accurately described by parametric families of probability distributions.

A related subject that is relevant to our present work concerns the use of filter statistics for representing subclasses of images. Zhu and Mumford [22] showed in their seminal work impressive image restoration results by employing filter statistics. A significant difference of their approach to established denoising methods using
Fig. 1. TV-Denoising fails for non-smooth image structure. A grid image superimposed with noise, shown left as image $d$, and three minimizers $u(\alpha)$ of $\|u - d\|^2 + \alpha \text{TV}(u)$ for increasing values of $\alpha$, computed with Chambolle’s projection algorithm [5]. Without prior knowledge about image structure, denoising is not possible. The approach introduced in this paper exploits filter statistics as prior knowledge for discriminating structure and noise through image approximation with level-set constraints – see Figs. 5 and 8.

TV-regularization [17] as well as to modern related approaches to image decomposition [5], [3], is the ability to generate image structure which is not possible when using the latter convex variational models (Fig. 1).

The nonconvex approach of Mumford and Shah involves two phases, learning and inference. In the learning phase, a Gibbs distribution

$$p(u) = \frac{1}{Z} \exp(-E(u)), \quad (1)$$

defined on the image space $\mathbb{R}^n$ together with coarsely quantized coordinates, is determined by maximizing the entropy of $p$ subject to the constraint that samples $u \sim p(u)$ reproduce given filter statistics on the average [23]. Stochastic sampling was applied to cope with this difficult optimization problem. Inference, on the other hand, amounts to approximate given image data $d$ by a function $u$ using the learned energy function $E(u)$ as a regularizing term. Because the latter is highly nonconvex, stochastic sampling was applied, too, for computing a local minimizer [22].

The objective of this paper is to present a quite different variational approach to solving a similar constrained approximation problem. Rather than encoding the filter statistics as prior information by a probability distribution that has to be learned beforehand, we bypass this entire learning step and exploit the prior knowledge directly by imposing hard constraints on the sizes of level-sets of filter outputs. Furthermore, we devise a deterministic algorithm that computes an approximation by solving a sequence of convex programs. The main purpose is the ability to study directly the formation of image structure through approximation in image spaces constrained by empirical distributions. Another aspect is that an explicit representation of the statistical knowledge is used for data processing, enabling to replace or to revise this knowledge quickly if an overall task is calling for, or if novel data are observed. This is not possible, however, when the available knowledge is implicitly represented by a probability distribution (1) defined over the entire image space, that has to be determined by a time-consuming prior learning process.

Our approach also allows to weaken these constraints in order to cope with uncertain prior knowledge – see Remark 2.2. We do not exploit this modification in the present paper, however.