Abstract  In this paper, we consider the resolution of constraint satisfaction problems in the case where the variables of the problem are subsets of $\mathbb{R}^n$. In order to use a constraint propagation approach, we introduce set intervals (named i-sets), which are sets of subsets of $\mathbb{R}^n$ with a lower bound and an upper bound with respect to the inclusion. Then, we propose basic operations for i-sets. This makes possible to build contractors that are then used by the propagation to solve problem involving sets as unknown variables. In order to illustrate the principle and the efficiency of the approach, a testcase is provided.

Keywords  Constraint propagation · Constraint satisfaction · Contractors · Interval analysis · Set intervals

Mathematics Subject Classification (2000)  62F30 Inference under constraints

1 Introduction

Constraint satisfaction problems involving subsets of $\mathbb{R}^n$ (namely set-valued constraint satisfaction problems or SVCSP for short) can appear in several engineering applications, typically, when arbitrary shapes (i.e. that cannot be parametrized) are involved. The reconstruction of a three dimensional object from photos [4], mapping an environment from sonar measurements [16, 20], SLAM (simultaneous localization and mapping) [11] or characterizing invariant sets of dynamic systems [2] can be represented by SVCSP. This paper introduces in Sect. 2 a new type of numbers, namely set intervals (or i-sets), which make possible to use constraint propagation methods for
solving SVCSP. Some basic operators for i-sets are also proposed. These operators are then used to build contraction operators (or contractors) in Sect. 3. An illustrative application is provided in Sect. 4 where a SVCSP is solved. Section 5 concludes the paper.

2 Set intervals (or i-sets)

2.1 Definition

Given two sets $A^-$ and $A^+$ of $\mathbb{R}^n$, the pair $[A^-, A^+]$ which encloses all sets $A$ such that

$$A^- \subset A \subset A^+$$

is a set interval (or i-set for short) and will be denoted by $[A]$ (see Fig. 1). The i-set $[\emptyset, \emptyset]$ is a singleton which contains a single element: the empty set $\emptyset$. The i-set $[\emptyset, \mathbb{R}^n]$ encloses all sets of $\mathbb{R}^n$. If $A^- \not\subset A^+$, then $[A^-, A^+]$ is empty. A i-set is a way to handle and to compute with uncertain sets (see [9,23]). The idea that is developed in this paper follows the foundations of interval analysis that has been built to handle uncertain real numbers [14,17], to solve real-valued nonlinear problems (see e.g. [7,10]), to minimize nonconvex criteria (see, e.g., [12,18]) or to provide mathematical proofs (see, e.g., [8,15,19,21]).

2.2 Operations

We shall now define some operations that can be used for i-sets. Two types of operations can be considered.

- **Specific i-set operations.** Since i-sets are sets (their elements are sets), the intersection, the union, the inclusion can be defined. In order to avoid any confusion with the operations of their elements, these operations will be denoted in a squared manner (e.g. $\cap$, $\cup$, $\in$).

- **Set extension.** All operations existing for elements of a i-set (which are sets) such as $\cap$, $\cup$, $\setminus$, $+$, reciprocal image , direct image, ... can be extended to i-sets [13].

Let us first start with specific i-set operations.

**Intersection** The i-set intersection between two i-sets is defined by

$$[A] \cap [B] = \{ X, X \in [A] \text{ and } X \in [B] \}.$$