A Proof of the Consistency of the Finite Difference Technique on Sparse Grids

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Abstract

In this paper, we give a proof of the consistency of the finite difference technique on regular sparse grids [7, 18]. We introduce an extrapolation-type discretization of differential operators on sparse grids based on the idea of the combination technique and we show the consistency of this discretization. The equivalence of the new method with that of [7, 18] is established.

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1. Introduction

Consider an univariate scaling function \( \phi \) which induces a multi-resolution analysis \( V^l \subset V^{l+1} \subset \cdots \) of \( L_2(\mathbb{R}) \), see [4], such that

\[
\bigcup_{l \in \mathbb{N}} V^l = L_2(\mathbb{R}), \quad V^l = \text{span}\{ \phi(2^l x - s) \mid s \in \mathbb{Z} \}.
\]

The wavelets \( \psi_{(l,t)} \) are the basis functions of the complementary spaces \( W_l \):

\[
W_l \oplus V^{l-1} = V^l, \quad W_l = \text{span}\{ \psi_{(l,t)} \mid t \in \mathbb{Z} \}.
\]

In this paper, we consider scaling functions and wavelets of the interpolent family which is introduced in Section 2. A simple way to the multivariate case is to use (in general anisotropic) tensor product wavelets, i.e.

\[
\psi_{(l,t)}(x) := \psi_{(l_1,t_1)}(x_1) \cdots \psi_{(l_d,t_d)}(x_d)
\]

(1)

where \( x = (x_1, \ldots, x_d) \), \( l = (l_1, \ldots, l_d) \), and \( t = (t_1, \ldots, t_d) \). Besides its simplicity this approach has a remarkable advantage with respect to its approximation power. Under some mild assumptions on the 1D wavelets there hold estimates of the type

\[
\inf_{u_h \in \text{span}_{l \leq \|l\|_1}} \| u - u_h \|_0 \leq C 2^{-\|l\|_1} \left\| \frac{\partial^{\|l\|_1}}{\partial x_1 \cdots \partial x_d} u \right\|_0 \quad \text{where} \quad \|l\|_1 := \sum_{i=1}^d l_i.
\]

(2)
If \( u \) has a compact support or decays sufficiently fast, then only \( O(2^n n^{d-1}) \) wavelets in \( \bigoplus_i W_i \) are necessary for the representation of \( u_h \). In contrast to this, \( O(2^m) \) degrees of freedom are required to achieve a similar accuracy if a single scale space \( V^1 := V^{l_1} \otimes \cdots \otimes V^{l_d} \) is used for the approximation. Standard references on this topic are [6, 9, 21].

In the last years, some techniques were developed to exploit the power of sparse grids for the solution of PDEs. A quite simple approach is to use the wavelets of (1) as trial functions for a Galerkin discretization [1, 2, 23]. Another approach is to use the tensor product wavelets as some sort of trial functions and to discretize differential operators by means of finite differences (FD) [7, 12, 18]. For this approach, consistency and stability are not easy to prove. Consistency was considered in [18] for the special case of so-called regular sparse grids where the trial space is \( \bigoplus_{\lambda \in \Lambda} W_i \). Here, \( \Lambda \) is a given finite index set, for example \( \Lambda = \{1 \mid |i|_1 \leq n\} \). As far as we know, there exists no proof of the stability of the discretization for general index sets \( \Lambda \). Nevertheless, there is strong numerical evidence for the stability, see [18]. The advantage of FD on sparse grids is the relatively simple evaluation of discrete differential operators, when compared to the Galerkin approach. Furthermore, the FD technique is more flexible, if special properties (e.g. mass conservation) have to be included into the discretization.

There is a third way to exploit the idea of sparse grids: The so-called combination technique [3, 8, 11, 14, 15, 16, 17] extrapolates a very accurate solution on \( \bigoplus_{\lambda \in \Lambda} W_i \) from solutions (FD or Galerkin) on subspaces \( V^1, i \in \Lambda \).

In this paper, we show the equivalence of the FD discretization of differential operators on sparse grids and the extrapolation formula of the combination technique applied to the discretized differential operators on the rectangular subgrids which correspond to \( V^1 \). We will use this result to give a new proof of the consistency of the sparse grid FD technique which improves previous results [18] in the following points:

- We need weaker smoothness requirements.
- The proof allows for quite general index sets \( \Lambda \).
- Higher order discretizations and even compact FD schemes are covered.

This paper is organized as follows. In Section 2 we briefly describe interpolol wavelets. We deal with the interpolation error associated to the interpolol approximation and the consistency error of general FD schemes. In Section 3 an extrapolation-type discretization on sparse grids is introduced and estimates for the consistency error are given. In the last section we show that this extrapolation scheme is just another form of the FD method on sparse grids.

### 2. Prerequisites

The basic properties and the efficiency of our sparse grid FD technique essentially rely on the simplicity of the underlying 1D wavelets, i.e. of the interpolols [5]. For our purpose it is convenient to consider interpolols of order \( N \) on the interval \([0, 1]\).