A New Multisection Technique in Interval Methods for Global Optimization

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Abstract

A new multisection technique in interval methods for global optimization is investigated, and numerical tests demonstrate that the efficiency of the underlying global optimization method can be improved substantially. The heuristic rule is based on experiences that suggest the subdivision of the current subinterval into a larger number of pieces only if it is located in the neighbourhood of a minimizer point. An estimator of the proximity of a subinterval to the region of attraction to a minimizer point is utilized. According to the numerical study made, the new multisection strategies seem to be indispensable, and can improve both the computational and the memory complexity substantially.

AMS Subject Classifications: 65K05, 90C30.

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1. Introduction

This paper investigates new multisection variants of a branch-and-bound algorithm [5, 8] for solving the box constrained global optimization problem [4, 8]:

$$\min_{x \in X} f(x),$$

(1)

where the $n$-dimensional interval $X \subseteq \mathbb{R}^n$ is the search region, and $f(x) : X \subset \mathbb{R}^n \to \mathbb{R}$ is the objective function. The global minimum value of $f$ is denoted by $f^*$, and the set of global minimizer points of $f$ on $X$ by $X^*$.

Herein real numbers are denoted by $x, y, \ldots$, and real bounded and closed interval vectors by $X = [X, X], Y = [Y, Y], \ldots$, where $X_i = \min X_i$ and $X_i = \max X_i$, for $i = 1, 2, \ldots, n$. The set of compact intervals is denoted by $\mathbb{I} := \{[a, b] | a \leq b, a, b \in \mathbb{R}\}$ and the set of $n$-dimensional interval vectors (also called boxes) by $\mathbb{I}^n$. A function $F : \mathbb{I}^n \to \mathbb{I}$ is called inclusion function of $f$ in $X \subset \mathbb{R}^n \to \mathbb{R}$, if $x \in X$ implies $f(x) \in F(X)$. In other words, $f(X) \subseteq F(X)$, where $f(X)$ is the range of the function $f$ on $X$. It is assumed in the present study that the inclusion function of the objective function is available (possibly given by interval arithmetic [5, 8]).
The width of the interval $X$ is defined by $w(X) = \overline{X} - \underline{X}$, if $X \in \mathbb{I}$, and $w(X) = \max_{i=1}^n w(X_i)$, if $X \in \mathbb{I}^n$. The midpoint of the interval $X$ is defined by $m(X) = (\overline{X} + \underline{X})/2$, if $X \in \mathbb{I}$, and $m(X) = (m(X_1), m(X_2), \ldots, m(X_n))^T$, if $X \in \mathbb{I}^n$. $F$ is said to be an inclusion isoton function over $X$ if $\forall Y, Z \in \mathbb{I}^n (X) Y \subseteq Z$ implies $F(Y) \subseteq F(Z)$. $F$ is called an order $\alpha$ (or $\alpha$-convergent) inclusion function of $f$ over $X$ if $\forall Y \in \mathbb{I}^n (X) w(F(Y)) - w(f(Y)) \leq Cw(Y)^\alpha$, where $C$ and $\alpha$ are some positive constants. We study the basic algorithm described in [3] but only the Cut-Off tests is used as the accelerating devices (Algorithm 1).

**Algorithm 1.** Basic B&B algorithm with midpoint and CutOff tests for Global Optimization. The notation “+” is used for entering and “−” for discarding elements in the tree.

```
1 proc GlobalOptimize(X,f,F,ε,T,Q) ≜
2     Q = {}                     Final Tree
3     T := (X, F(X))            Work Tree
4     ˜f = F(X)                 Upper bound for $f^*$
5     while (T ̸= {})
6         (X, F(X)) := \{(X, F(X)) ∈ T | F(X) = \min\{F(X_i), \forall (X_i, F(X_i)) ∈ T\}
7         T := T - (X, F(X))       Remove (X, F(X)) from T
8         ˜f = \min\{ ˜f, ˜f(X) \} Improve upper bound for $f^*$
9         Subdivide(X, U_1, \ldots, U_s)
10        for i := 1 to s
11            F(U_i) := F(U_i)
12            if ( ˜f < F(U_i))
13                next_i
14                if (w(U_i) < ε)
15                    Q := Q + (U_i, F(U_i)) Store U_i and F(U_i) in Q
16                else T := T + (U_i, F(U_i)) Store U_i and F(U_i) in T
17            T := CutOffTest(T, ˜f) \forall (X, F(X)) ∈ T, ˜f < F(X) ⇒ remove (X, F(X)) from T
18        end
```

2. **Multisection in Global Optimization**

Now we investigate the following ways of box subdivision. The first one (a) is the traditional bisection by the widest interval component [8]. According to new studies [1, 3, 5, 7], multisection may improve the efficiency of such branch-and-bound techniques. The second way of subdivision, (b) provides $2^n$ subintervals by halving all interval components. The subdivision techniques (c) and (d) result in $3^n$ and $4^n$ subintervals, respectively. The new subdivision method will use three parameters, $P_1$, $P_2$ and $pf$ and depending on the relative value of these parameters