Gradient Recovery for Singularly Perturbed Boundary Value Problems I: One-Dimensional Convection-Diffusion

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Abstract

We consider a Galerkin finite element method that uses piecewise linears on a class of Shishkin-type meshes for a model singularly perturbed convection-diffusion problem. We pursue two approaches in constructing superconvergent approximations of the gradient. The first approach uses superconvergence points for the derivative, while the second one combines the consistency of a recovery operator with the superconvergence property of an interpolant. Numerical experiments support our theoretical results.

AMS Subject Classifications: 65L10, 65L60.

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1. Introduction

Let us consider the singularly perturbed boundary value problem

\[-\varepsilon u'' - b(x)u' + c(x)u = f(x) \quad \text{in } (0, 1), \quad u(0) = u(1) = 0, \quad (1.1)\]

where $0 < \varepsilon \ll 1$ is a small parameter; $b$, $c$ and $f$ are smooth functions with $b(x) > 1$ for $x \in [0, 1]$. Without loss of generality we shall also assume that $(b'/2 + c)(x) > 0$ on $[0, 1]$, which can be achieved by a simple transformation.

Then the solution $u$ of (1.1) typically has an exponential boundary layer of width $\mathcal{O}(\varepsilon \ln 1/\varepsilon)$ at $x = 0$.

It is nowadays well-known that the finite element method on layer-adapted meshes leads to error estimates that are uniform with respect to the perturbation parameter $\varepsilon$. In the current paper we shall consider a class of Shishkin-type meshes as introduced in [12]; see Section 2.

For instance, for the linear Galerkin FEM on Shishkin’s original piecewise uniform mesh with $N + 1$ mesh points, the following error estimate in the $\varepsilon$-weighted $H^1$ norm holds true:

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\[ |||u - u_N||_e \leq CN^{-1} \ln N, \quad (1.2a) \]

where \[ |||w||_e^2 = \varepsilon ||u' - u'_N||_0^2 + ||u - u_N||_0^2 \] is the \( L_2(0, 1) \) norm, and \( u_N \) is the computed solution. Here and throughout the paper \( C \) denotes a generic positive constant that is independent of \( \varepsilon \) and of the mesh parameter \( N \). The error bound (1.2a) was proved (even for the two-dimensional case) in [15]. For the central difference approximation to (1.1) – which can be regarded as a version of the linear Galerkin FEM with approximate evaluation of the integral expressions involved – we have the global maximum-norm error estimate

\[ ||u - u_N||_\infty \leq C(N^{-1} \ln N)^2. \quad (1.2b) \]

This is a consequence of the nodal error estimates in, e.g., [1, 7] and the interpolation error estimates of [3].

In 1998/99 the first superconvergence results for (1.1) and its two-dimensional analogue were published [9, 14, 18]. Originally the term “superconvergence” was used if the rate of convergence at some exceptional points exceeds the optimal global rate [17]. If the convergence rate in a discrete (semi-) norm say \[ || \cdot \cdot ||_d,e \] exceeds the rate of convergence in its continuous counterpart \[ || \cdot \cdot ||_e \] this phenomenon is called superconvergence, too. In our paper we shall also use the concept of superconvergent recovery operators as described in [6].

Let \( u' \) denote the (bi)linear interpolant of \( u \) on a Shishkin mesh. In [9] the author proved for the Galerkin FEM in 2D that uses bilinear elements on rectangles

\[ |||u' - u'_N||_e \leq CN^{-2}(\ln N)^{5/2}. \quad (1.3) \]

This is a superconvergence result because in the continuous norm the order of convergence is smaller, see (1.2a). As a consequence of (1.3) we have the almost-optimal \( L_2 \)-error estimate

\[ ||u - u_N||_0 \leq CN^{-2}(\ln N)^{5/2}. \]

As pointed out in, e.g. [2], superconvergence properties like (1.3) are basic ingredients for the superconvergent recovery of gradients. Furthermore, if a superconvergent recovery operator is available, then it is possible to define an a posteriori error estimator that is asymptotically exact.

In the current paper we study computable superconvergent approximations of the weighted gradient of the solution of problem (1.1) on Shishkin-type meshes. Up to now, we do not know of any result in this direction from the literature. For the unperturbed case we refer the reader to the surveys [2, 5, 17].

2. Shishkin-Type Meshes and Solution Decomposition

For our discretization we use a mesh of the general type introduced in [12]. Let \( \tau \) denote a mesh transition parameter defined by