H. M. Ma · X.-L. Gao

Eshelby’s tensors for plane strain and cylindrical inclusions based on a simplified strain gradient elasticity theory

Abstract The Eshelby tensor for a plane strain inclusion of arbitrary cross-sectional shape is first presented in a general form, which has 15 independent non-zero components (as opposed to 36 such components for a three-dimensional inclusion of arbitrary shape). It is based on a simplified strain gradient elasticity theory that involves one material length scale parameter. The Eshelby tensor for an infinitely long cylindrical inclusion is then derived using the general form, with its components obtained in explicit (closed-form) expressions for the two regions inside and outside the inclusion for the first time based on a higher-order elasticity theory. This Eshelby tensor is separated into a classical part and a gradient part. The latter depends on the position, the inclusion size, the length scale parameter, and Poisson’s ratio. As a result, the new Eshelby tensor is non-uniform even inside the cylindrical inclusion and captures the size effect. When the strain gradient effect is not considered, the gradient part vanishes and the newly obtained Eshelby tensor reduces to its counterpart based on classical elasticity. The numerical results quantitatively show that the components of the new Eshelby tensor vary with the position, the inclusion size, and the material length scale parameter, unlike their classical elasticity-based counterparts. When the inclusion radius is comparable to the material length scale parameter, it is found that the gradient part is too large to be ignored. In view of the need for homogenization analyses of fiber-reinforced composites, the volume average of the newly derived Eshelby tensor over the cylindrical inclusion is obtained in a closed form. The components of the average Eshelby tensor are observed to depend on the inclusion size: the smaller the inclusion radius, the smaller the components. However, as the inclusion size becomes sufficiently large, these components are seen to approach from below the values of their classical elasticity-based counterparts.

1 Introduction

Eshelby’s equivalent eigenstrain method and solution for the ellipsoidal inclusion problem [1,2] are cornerstones of micromechanics (e.g., [3,4]). Eshelby’s fourth-order strain transformation tensor is essential for homogenization methods including the Mori–Tanaka method (e.g., [5–7]) and the self-consistent method (e.g., [8–11]). However, Eshelby’s original formulation [1,2] is based on classical elasticity, and the resulting Eshelby tensor for an ellipsoidal inclusion depends only on Poisson’s ratio of the material and the aspect ratio of the inclusion (rather than its actual size). Consequently, homogenization methods developed using the classical elasticity-based Eshelby tensors cannot account for the inclusion (particle) size effect on elastic properties (e.g., [12]), which has been experimentally observed to exist in polymer matrix composites filled by micron- and nano-sized particles (e.g., [13–16]). Hence, efforts have been made to study the Eshelby inclusion problems and to obtain Eshelby’s tensors using higher-order elasticity theories, which contain material length
scale parameters and can explain the size-dependent elastic deformations of polymers (e.g., [17]). It is believed that the size effect exhibited by some elastically deformed polymers is related to Frank elasticity resulting from small but non-zero bending stiffness of polymer chains and their non-negligible non-local interactions.

The higher-order elasticity theories that have been applied to study the Eshelby inclusion problems include a micropolar theory [18–20], a microstretch theory [21–23], a modified couple stress theory [24], and a simplified strain gradient theory [25]. Closed-form solutions obtained in these studies are mostly for the problem of a spherical inclusion. For the problem of an infinitely long cylindrical inclusion embedded in an infinite elastic body and subjected to a prescribed eigenstrain, an analytical study was presented in Cheng and He [19] using a micropolar theory, and the average Eshelby tensor was given as the result of a special case in Ma and Hu [20,23], where the Eshelby ellipsoidal inclusion problem was studied using a micropolar theory and a microstretch theory, respectively. However, the micropolar theory used in Cheng and He [19] and Ma and Hu [20] contains four material parameters in addition to the two Lamé constants, and the microstretch theory employed in Ma and Hu [23] involves seven additional material parameters. Due to the difficulties in determining these microstructure-dependent material length scale parameters (e.g., [26,27]) and in dealing with fourth-order Eshelby tensors, it is very desirable to solve the Eshelby cylindrical inclusion problem using a higher-order elasticity theory containing only one additional material length scale parameter.

The current paper aims to provide such a solution for the Eshelby cylindrical inclusion problem, which is closely related to fiber-reinforced composites (e.g., [28]). The solution is derived in a closed form, and the Eshelby tensor for the two regions inside and outside the cylindrical inclusion is obtained in explicit expressions for the first time using a higher-order elasticity theory. The formulation is based on a simplified gradient strain elasticity theory that involves only one material length scale parameter in addition to the two classical elastic constants [29]. This theory has recently been used by Gao and Ma [25] to obtain a general form of the Eshelby tensor having 36 independent components and to derive the closed-form expression of the Eshelby tensor for a spherical inclusion. The present study continues the work started in Gao and Ma [25] and extends it to plane strain and cylindrical inclusion problems.

The rest of this paper is organized as follows. In Sect. 2, the Eshelby tensor for a plane strain inclusion of arbitrary cross-sectional shape is presented in a general form, which has 15 independent components. In Sect. 3, closed-form expressions for the Eshelby tensor inside and outside the inclusion are derived for a cylindrical inclusion, and the volume average of the Eshelby tensor over the cylindrical inclusion is exactly determined. Numerical results are provided in Sect. 4 to quantitatively illustrate the position dependence and the inclusion size dependence of the newly obtained Eshelby tensor for the cylindrical inclusion. The paper concludes with a summary in Sect. 5.

2 Eshelby tensor for a plane strain inclusion

In the simplified strain gradient elasticity theory (e.g., [29]), the Navier-like displacement equations of equilibrium have the following form:

\[
(\lambda + \mu)u_{i,ij} + \mu u_{j,ik} - L^2 \left[ (\lambda + \mu)u_{i,ij} + \mu u_{j,ik} \right]_{,mm} + f_j = 0,
\]

where \( u_i \) is the displacement component, \( f_j \) is the body force component, \( \lambda \) and \( \mu \) are the Lamé constants, and \( L \) is the material length scale parameter (with \( L^2 = c \), \( c \) being the strain gradient coefficient used in Gao and Park [29]). Clearly, Eq. (1) reduces to the Navier equations in classical elasticity when \( L = 0 \) (i.e., when the strain gradient effect is not considered).

Note that the standard index notation, together with the Einstein summation convention, is used in Eq. (1) and throughout this paper, with each Latin index (subscript) ranging from 1 to 3 and each Greek index ranging from 1 to 2 unless otherwise stated.

By solving Eq. (1) subject to the boundary conditions of \( u_i \) and its derivatives vanishing at infinity through using Fourier transforms, the Green’s function in the simplified strain gradient theory has been derived in terms of elementary functions in Gao and Ma [25]. The use of the Green’s function method then leads to the determination of the Eshelby tensor for an inclusion of arbitrary shape based on the simplified strain gradient elasticity theory. Note that in solving the Eshelby inclusion problem using the Green’s function method the difference in the boundary conditions between the classical case obeying the Navier equations and the current strain gradient case governed by Eq. (1) is that for the former only the displacement components and their first derivatives are required to vanish at infinity, while for the latter the second and third derivatives of the displacement components need to vanish at infinity additionally.