Benoit Panicaud

Application of Clifford algebra $C\ell_3(\mathbb{C})$ to continuum and engineering mechanics

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Abstract Multivectorial algebra is of both academic and technological interest. Its application, however, is not always easy. A distinction must be made between polar and axial vectors and between scalars and pseudo-scalars. Eight element types are often considered even if they are not always identified as multivectors. In some cases, for simplicity’s sake, only vectorial algebra or quaternion algebra is explicitly used for physical and mechanical applications. It would, however, be more convenient to use more complex algebra directly in order to have a wider range of mechanical applications. The aim of this paper is to examine one particular type of Clifford algebra that could solve this problem. The present study focuses on showing how these quantities can be used to model mechanical and engineering problems. First, continuum mechanics in a Cauchy medium is investigated for elastic transformations. Second, a specific type of shot-peening application is studied. Applications are then used to illustrate the scope and efficiency of this type of modeling based on geometric algebra.

1 Introduction

Since their discovery by William Rowan Hamilton in 1843, quaternions (\mathbb{H}) have been the subject of many publications [1–3]. Quaternions form the smallest associative division algebra [4] and have led to several applications in mechanical engineering in relation to space rotation group SO(3), such as robotics [5] or the motion of space shuttles and satellites [6]. Other applications concern chemical problems and crystallography [7, 8]. However, there is another way of obtaining “new” numbers. Instead of working on the number’s properties, we can work on the associated algebraic structure. Based on this premise, William Kingdon Clifford introduced the well-known Clifford algebra, noted $C\ell_{p,q}$, in 1843 [9]. Many papers have discussed Clifford algebra and/or its applications, mainly in relation to physical problems [10–12]. Some of them deal with $C\ell_{0,3}$ algebra in \mathbb{R} or with a complexified version, noted $C\ell_{3}(\mathbb{C})$ [13–15]. In this paper, we will investigate the possible applications of this type of algebra, especially for mechanical problems.

We will start with the hyperbolic (or split complex) form of biquaternion algebra isomorphic to $C\ell_{0,3}$, and since this form is not satisfactory for relativity and space-time quantities, it may need to be complexified, resulting in $C\ell_{3}(\mathbb{C})$ algebra. It is therefore not the best choice as far as a mathematical framework is concerned because it means avoiding the classical use of $C\ell_{1,3}$, which would have been a quicker starting point, but would have resulted in a presentation that is fully relativistic, but quite useless for the transformations investigated in this paper. This type of algebra has been already used successfully for general field theories and is described in [16].

B. Panicaud (✉)
Laboratoire des Systèmes Mécaniques et d’Ingénierie Simultanée (LASMIS), Institut Charles Delaunay (ICD), CNRS UMR 6279, Université de Technologie de Troyes (UTT), 12, rue Marie Curie, P.O. Box 2060, 10010 Troyes cedex, France
E-mail: benoit.panicaud@utt.fr
Tel.: +33-3-25718061
Fax: +33-3-25715675
The present article starts with a definition of the Clifford algebra being considered. It goes on to give a synthetic presentation of biquaternions that is of practical interest. It is important to emphasize that this has already been attempted by means of other mathematical groups or numbers, especially sedenions by Köplinger [17–19], and hyperbolic numbers by Ulrych [20–22]. All Ulrych’s work could obviously be produced using $C\ell_3$ algebra, as shown by [23,24] in their publication on electrodynamics; it could also be produced using complex quaternions $C\ell(\mathbb{C})$ as demonstrated by many others, as far back as [4,25]. However, with the exception of Hestenes’ publication, these authors usually deal with relativistic transformations and/or electromagnetic and gravitational interactions (see in [26–28]), and to our knowledge, none of them concern mechanical problems (except robotics which is not the aim of the present paper). The objective of this paper is therefore to propose possible applications of this generalized algebra but for more conventional cases. Particular applications are investigated. First, continuum mechanics is studied through the elementary decomposition of general propositions are investigated. First, continuum mechanics is studied through the elementary decomposition of general propositions are investigated.