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A bifurcated waveguide problem

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Abstract We consider the diffraction of the dominant mode in a parallel-plate waveguide with hard boundaries which is incident on a centered soft-hard half-plane. By using the Fourier transform technique in conjunction with the Mode Matching method, the related boundary value problem is formulated as a modified Wiener–Hopf equation. The solution of the latter, which contains infinitely many constants, is found by solving numerically an infinite set of linear algebraic equations.

Key words Bifurcated waveguide · Propagation · Mixed boundary value problems · Wiener–Hopf technique · Mode matching method

1 Introduction

Parallel-plate waveguide bifurcation problems have been subject to numerous past investigations due to their importance in acoustic and electromagnetic wave propagation theory. The results of earlier works, which can be found in the “Waveguide Handbook” (Marcuvitz 1964) were mostly obtained through the Wiener–Hopf technique. However some alternative methods are also available. For example, Hurd and Gruenberg (1954) used the “Residue Calculus technique” to solve the Dirichlet problem while Mittra (1963) used the “direct inversion” of the doubly infinite set of equations which arise.

In this work, we consider the diffraction of the dominant mode in a parallel-plate rigid waveguide, which is incident on a centered half-plane with soft and hard boundary conditions holding on its top and bottom faces (Fig. 1). Two similar waveguide bifurcation problems have been considered recently by Lüneburg and Hurd (1981) in the cases where the two narrow waveguides are characterized by soft-hard-soft-hard and by soft-soft-hard-hard boundary conditions, respectively.

The classical formulation of a problem similar to that considered in the present work leads to a matrix Wiener–Hopf equation where the solution requires splitting a square matrix into the product of two matrices with nonvanishing determinants such that these matrices as well as their inverses are regular and of algebraic growth in certain overlapping halves of a complex plane. Unfortunately, no known method exists to perform the Wiener–Hopf factorization of the kernel matrix related to the present problem. In (1a), \(d\) is the distance between the parallel rigid plates and the centered half-plane as shown in Fig. 1, \(K(z)\) is the square-root function given by

\[
\begin{bmatrix}
1 & \cotan[Kd] \\
-\frac{K}{\cotan[Kd]} & 1
\end{bmatrix}
\]

(1a)

related to the present problem. In (1a), \(d\) is the distance between the parallel rigid plates and the centered half-plane as shown in Fig. 1, \(K(z)\) is the square-root function given by

\[K(z) = \sqrt{k^2 - z^2},\]  

(1b)

while \(z\) is the Fourier transform variable, and \(k\) is the free-space wave number.

To overcome this difficulty, in this work an alternative formulation consisting of employing the Mode Matching method in conjunction with the Fourier transform technique will be adopted. This hybrid formulation will yield a single (scalar) Wiener–Hopf equation involving an infinite system of linear algebraic equations which can be solved rather easily by means of numerical procedures. It is to be noted that a variant of this method was first applied by Matsui (1965) and Ando (1969/1970) for the

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problem of diffraction of sound waves by a semi-infinite cylindrical rigid tube of certain wall thickness, and later by Yoshidomi and Aoki (1988) and Büyüksüsoy and Polat (1998a, b) in treating the scattering of electromagnetic waves by some impedance structures.

A time factor $e^{-j\omega t}$ with $\omega$ being the angular frequency is assumed and suppressed throughout the paper.

### 2 Formulation of the problem

We consider the diffraction of the incident acoustic wave

$$u'(x, y) = e^{ikx}, \quad y \in (-d, d)$$

from the bifurcated waveguide shown in Fig. 1. It is obvious that $u'$ would consist of the dominant mode which could propagate if the inner half-plane was absent. The bifurcated waveguide is formed by two infinite parallel rigid plates located at $y = d$ and $y = -d$, respectively, and a centered half-plane, defined by $\{(x, y, z) | x > 0, y = 0, z \in (-\infty, \infty)\}$, which is soft at the top and hard at the bottom.

For the sake of analytical convenience we shall assume that $k$ possesses a positive imaginary part. The results related to lossless case can be obtained by making $\Im m(k) \to 0$ at the end of the analysis.

An expression of the total field will be established as follows:

$$u^T(x, y) = \begin{cases} [u'(x) + u_1(x, y)]H(-x) + u_2(x, y)H(x), & 0 < y < d \quad (3) \\ u'(x) + u_3(x, y), & -d < y < 0 \end{cases}$$

where $u'$ is the incident field given by (2), and $H(x)$ denotes the Heaviside unit step function. $u_j, j = 1, 2, 3$, which satisfy the Helmholtz equation, are to be determined with the aid of the following boundary and continuity relations:

$$\frac{\partial}{\partial y} u_1(x, d) = 0, \quad x > 0 \quad (4a)$$

$$u_2(x, 0) = 0, \quad x > 0 \quad (4c)$$

$$\frac{\partial}{\partial y} u_3(x, 0) = 0, \quad x > 0 \quad (4d)$$

$$\frac{\partial}{\partial y} u_3(x, -d) = 0, \quad -\infty < x < \infty \quad (4e)$$

$$u_1(x, 0) - u_3(x, 0) = 0, \quad x < 0 \quad (4f)$$

$$\frac{\partial}{\partial y} u_1(x, 0) - \frac{\partial}{\partial y} u_3(x, 0) = 0, \quad x < 0 \quad (4g)$$

$$u'(0) + u_1(0, y) = u_2(0, y), \quad 0 < y < d \quad (4h)$$

$$\frac{\partial}{\partial y} u^T(x, 0) = O(x^{-3/4}). \quad (5b)$$

### 3 Solution of the problem

3.1 The region $y \in (-d, 0)$

Consider first the region $y \in (-d, 0)$ where $u_3(x, y)$ satisfies the Helmholtz equation in the range $x \in (-\infty, \infty)$. Its Fourier transform with respect to $x$ gives

$$\left[\frac{d^2}{dy^2} + (k^2 - z^2)\right]F(x, y) = 0 \quad (6a)$$

with

$$F(x, y) = F_+(x, y) + F_-(x, y) \quad (6b)$$

and

$$F_\pm(x, y) = \pm \int_0^\infty u_3(x, y) e^{\pm i\lambda x} dx. \quad (6c)$$

By taking into account the asymptotic behaviors of $u_3$ for $x \to \pm \infty$, namely:

$$u_3(x, y) = O(e^{\pm ikx}), \quad x \to \pm \infty \quad (7)$$

one can show that $F_+(x, y)$ and $F_-(x, y)$ are regular functions of $z$ in the half-planes $\Im m(z) > \Im m(-k)$ and $\Im m(z) < \Im m(k)$, respectively.

The solution of (6a) satisfying (4e) reads

$$F_-(x, y) + F_+(x, y) = A(x) \cos [K(y + d)], \quad (8)$$