Arbitrage-free discretization of lognormal forward Libor and swap rate models

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Abstract. An important recent development in the pricing of interest rate derivatives is the emergence of models that incorporate lognormal volatilities for forward Libor or forward swap rates while keeping interest rates stable. These market models have three attractive features: they preclude arbitrage among bonds, they keep rates positive, and, most distinctively, they price caps or swaptions according to Black’s formula, thus allowing automatic calibration to market data. But these features of continuous-time formulations are easily lost when the models are discretized for simulation. We introduce methods for discretizing these models giving particular attention to precluding arbitrage among bonds and to keeping interest rates positive even after discretization. These methods transform the Libor or swap rates to positive martingales, discretize the martingales, and then recover the Libor and swap rates from these discretized variables, rather than discretizing the rates themselves. Choosing the martingales proportional to differences of ratios of bond prices to numeraire prices turns out to be particularly convenient and effective. We can choose the discretization to price one caplet of arbitrary maturity without discretization error. We numerically investigate the accuracy of other caplet and swaption prices as a gauge of how closely a model calibrated to implied volatilities reproduces market prices. Numerical results indicate that several of the methods proposed here often outperform more standard discretizations.

Key words: Interest rate models, Monte Carlo simulation, market models

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1 Introduction

A major development in the modeling of interest rates for pricing term structure derivatives is the emergence of models that incorporate lognormal volatilities for forward rates while keeping rates stable. It was noted by Heath, Jarrow, and Morton [12] that in the general class of models they developed based on continuously compounded forward rates, lognormal volatilities lead to rates that become infinite in finite time with positive probability. By working instead with various types of discretely compounded rates, Sandmann and Sondermann [23, 24], Brace et al. [5], Goldys et al. [10], Miltersen et al. [19], Musiela and Rutkowski [21], and Jamshidian [14, 15] have overcome this difficulty and developed well-posed models that indeed admit deterministic diffusion coefficients for the logarithms of forward rates — i.e., lognormal volatilities. The rates themselves are not simultaneously lognormal, but each becomes lognormal under an appropriate change of measure.

This class of models — often referred to as market models because of their consistency with market conventions — have three principal attractions:

- they preclude arbitrage among bonds (and just as in the HJM [12] framework this means that the drift is determined once the volatilities are specified);
- they keep rates positive (a consequence of the lognormal form of the volatility that further precludes arbitrage between bonds and cash);
- they price caplets or swaptions according to Black’s [3] formula, consistent with market practice.

The first property corresponds to what Musiela and Rutkowski [20] call a weak no-arbitrage condition and the first two together make up their full no-arbitrage condition. The last feature means that the models are easily calibrated to market data. Market participants quote caplet and swaption prices according to their Black implied volatility; if these implied volatilities are used as inputs to a market model, market prices are recovered exactly.

These attractive properties must, however, be understood as features of continuous-time models. (Though discretely compounded, the forward rates evolve continuously). Pricing complex path-dependent instruments in these models typically requires numerical computation and thus discretization. A casual discretization can easily lead to a model without any of the three attractive properties identified above. Since it is ultimately the discretized model that is used for pricing, the theoretical advantages of the continuous-time models are potentially lost in practice. The gap between the discretized and continuous models can be substantial because rather coarse time discretizations (e.g., with an increment of three months) are frequently used in practice.

This paper develops discretizations of lognormal forward Libor and forward swap rates that preserve some, though not all of the attractive features of the continuous-time formulations, and appear to be substantially better than naive discretizations in several respects. We put particular emphasis on discretizations