Complete markets with discontinuous security price

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Abstract. A parameterized family of financial market models is presented. These
models have jumps intrinsic to the price processes yet have strict completeness,
equivalent martingale measures, and no arbitrage. For each value of the parameter
\( \beta \) \((−2 < \beta < 0)\) the model is just as rich as the standard model using white
noise (Brownian motion) and a drift; moreover as \( \beta \) increases to zero the model
converges weakly to the standard model. A hedging result, analogous to the
Karatzas-Ocone-Li theorem, is also presented.

Key words: Market completeness, arbitrage, stochastic calculus, Azéma mar-
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tingale central limit theorem

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1 Introduction

A classical model for continuous time asset pricing models is of the form
\[
Y_t = 1 + \int_0^t \sigma Y_s d W_s + \int_0^t m Y_s ds,
\]
or geometric Brownian motion. We will call this the *standard model*. There is a
highly developed theory based on this model (see, for example [5]). Of course,
this model has been extended to much more generality: \( Y \) can be a diffusion or

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even an Itô process (see for example the recent monograph of Karatzas [12]). An important feature of the standard model is that markets are “complete”: that is, a contingent claim that is measurable with respect to the filtration generated by the stock price process is a redundant claim. (This means that there will always exist a self-financing trading strategy that replicates the contingent claim.) However as has been long known (see for example [18] or [10]), empirical studies show that stock prices often have jumps (that is, are not continuous). An interesting recent example concerning the term structure of interest rates and monetary policy, where jumps clearly occur, is given in [1].

Various models incorporating discontinuities have been proposed. The most common is the “jump diffusion” model, where Poisson jumps have been added to Brownian (or “white”) noise, modeled by a Wiener process. This has many advantages: the noise process $W_t + N_t$ is still a Lévy process (a process with stationary and independent increments) and thus can be justified by central limit type arguments; the solution of the corresponding stochastic differential equation (the stock price process) is a strong Markov process. These models are useful to model stock prices whose jumps arise from exogenous events (such as natural disasters, interest rate announcements, etc.; see [18], [1], or [11]), rather than to model those for which the jumps are intrinsic to the “trading noise”. These models do have equivalent martingale measures, and they do have relaxed market completeness in the sense described below in Definition 1.2 (see, e.g., [11] or [24]). Here we are interested in models that have strict market completeness: this allows one to use a single hedging strategy only, instead of having to use one for the continuous martingale noise and a separate one for the Poisson-type noise. This is especially valuable when it is believed the jumps are intrinsic to the market noise and do not arise from exogenous “shocks”.

In attempts to allow for more general models, recent work has been devoted to studying what can be done when a model does not have completeness, and what adjustments can be made in those cases. See, for example, [20], [22], [23].

In this paper we show that one can indeed have models that have market completeness and jumps that are intrinsic to the stock price with a unique equivalent martingale measure. To emphasize the intrinsic nature of the discontinuities in our model, we define two types of market completeness. Let $Y_t$ be the stock price process and let $\mathcal{F}_t = \sigma(Y_s; s \leq t)$ be its natural filtration, made right continuous and complete. If $Y$ for example is a locally bounded semimartingale, then the absence of arbitrage is equivalent to the existence of a risk-neutral measure, also known as an equivalent measure. That is, there exists another probability measure $P^*$ with the same null sets as $P$ such that $Y$ is a local martingale (see, e.g., Theorem 1.1 of [3, p.479]). For a contingent claim $H$ in $L^1(\mathcal{F}_T; dP^*)$, let $M_t^H = E^*[H \mid \mathcal{F}_t]$ be the uniformly integrable $P^*$ martingale closed by $H$.

**Definition 1.1.** A model has strict market completeness if for any contingent claim $H \in L^1(\mathcal{F}_T; dP^*)$ there exists a predictable process $\xi_t^H$ such that

$$M_t^H = E^*[H] + \int_0^t \xi_s^H \, dY_s,$$

where $\xi_t^H$ is predictable.