Introduction to a theory of value coherent with the no-arbitrage principle

Marco Frittelli

Department of Quantitative Methods in Economics, University of Milano - Bicocca, 20126 Milano, Italy (e-mail: Marco.Frittelli@unimib.it)

Abstract. This paper defines the value of a general claim based on agent’s preferences and coherent with the No Arbitrage Principle. This Value is a non trivial extension of the certainty equivalent since it takes into consideration the possibility of partially hedging the risk carried by the claim.

When the market is complete this Value is the unique no arbitrage price. When the risk may not even be partially covered, this Value is the certainty equivalent.

Between these two cases just some of the risk may be hedged and the no arbitrage principle requires the price to lie in the “arbitrage interval”. The Value we propose is exactly designed to satisfy this condition.

Key words: Certainty Equivalent, Asset Pricing, No Arbitrage, Equivalent Martingale Measure, Incomplete Market.

JEL Classification: G10, G12, D52, D46.

Mathematics Subject Classification (1991): 60G42, 60G44, 90A09, 90A10.

1 Introduction

1.1 Notations

A (not empty) family $\Sigma$ of adapted stochastic processes on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, P)$ represents price processes of the securities available for trading in the market. Set $\mathcal{F} = [0,T]$ and $\mathcal{F} = \mathcal{F}_T$.

One process, say $S^0 \in \Sigma$, is taken as numeraire. One may think that $S^0$ represents the money market account process. The deflator process is $B = (B_t)_{t \in \mathcal{F}}$.
\[ B_t = 1/S_t^0 \] and we assume that \( B \) is bounded and positive. We then consider the family \( \chi \) of deflated price processes \( \chi = \{ X = BS_t, S \in \Sigma \} \) and note that by construction \( \chi \) contains at least the constant process equal to 1.

We denote with \( \mathcal{M} \) the set of probability measures \( Q \) absolutely continuous with respect to \( P \) such that all processes in \( \chi \) are \((\mathcal{F}_t, Q)\)-martingales. We assume the existence of a martingale measure equivalent to \( P \). We will see that, in specific cases, we can replace this assumption with the weaker condition that \( \mathcal{M} \neq \emptyset \), since the existence of an equivalent martingale measure will be derived from other assumptions.

With \( u : \mathcal{D} \to \mathbb{R} \) we always denote a non decreasing real function defined on an interval \( \mathcal{D} \subseteq \mathbb{R} \) with nonempty interior and taking the value \(-\infty\) on the external points of \( \mathcal{D} \). We denote with \( L^-(\Omega, \mathcal{F}_T, P) \) the set of bounded below random variables. A \( T \)-claim \( w \) is an element of \( L^-(\Omega, \mathcal{F}_T, P) \).

### 1.2 A definition of value coherent with no arbitrage

In this introduction we present the main idea of the paper. We defer to the next section the precise formulation and the assumptions that guarantee existence and uniqueness of the Value of the claims. Here we assume that all integrability requirements are satisfied. For expositional reasons in this section we set \( B_T = 1 \) and assume that \( u \) is strictly increasing.

Let \( w \) be a time \( T \) claim that we want to evaluate. The function \( u \) is the time-\( T \) utility (i.e. utility from time \( T \) wealth).

- **Totally incomplete market**
  
  In this case the possibility of trading in the available marketed assets does not provide any help for hedging (not even partially) the risk carried by \( w \) (consider for example the market where the only traded asset has price equal to the constant 1). In this case the subjective value of \( w \) is traditionally assigned by the certain amount \( \pi(w) \in \mathbb{R} \) which utility is equal to the expected utility of the claim \( w \):  
  \[
  u(\pi(w)) = E_Q[u(w)].
  \]  
  The agent can’t take advantage of the presence of the marketed securities.

- **Complete market**
  
  If a bounded claim \( w \) is attainable by a self-financing strategy in the marketed assets or if the market is complete (the canonical example is the Black and Scholes market), the value of the claim is independent from agents preferences and it is univocally assigned by the formula:  
  \[
  \pi(w) = E_Q[w]
  \]
  where \( Q \) is any martingale measure (eventually unique if the market is complete).