Multigrid methods for discrete elliptic problems on triangular surfaces

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1 Introduction

Geometric differential equations play a crucial role in many applications ranging from material science to image processing or numerical relativity [10, 13]. Numerical discretizations of such problems typically lead to large algebraic systems which become ill-conditioned with decreasing mesh size. For example, the approximation of an evolving surface driven by mean curvature requires the solution of a second order elliptic problem on a triangular surface in each time step [6].

A straightforward approach to construct multigrid methods on triangular surfaces is to simply use the same weights for restriction and prolongation as for planar triangulations. Such algorithms have been implemented in the software package MC and applied successfully to the numerical solution of the Einstein equations [11]. Special coarsening strategies for unstructured triangular surface meshes have been considered in [1]. In spite of the simplicity of the straightforward approach, numerical experiments indicate multigrid convergence speed. However, there seems to be no theoretical justification yet. Biorthogonal wavelet bases on manifolds provide an essential step towards mesh independent preconditioners [5]. However, the construction and thus the resulting algorithms involve piecewise smooth parameterizations of the underlying manifold which might cause problems, if the manifold itself has been computed numerically.

In this paper, we provide a convergence analysis for a class of multigrid methods for discretized self-adjoint elliptic problems on a triangular surface $M_j$. As these multigrid methods involve the same weights for restriction and prolongation as for planar triangulations, our results can regarded as a theoretical justification of the above-mentioned straightforward approach. The main difficulty in the analysis is resulting from the fact that an underlying sequence $M_k$, $k = 1, \ldots, j$, of coarser triangular surfaces does no longer generate a sequence of nested finite element spaces. As a consequence, the existing convergence theory for subspace correction methods [14, 15] cannot be applied directly. A possible way out might be to make use of generalizations to non-nested spaces and varying forms [4, Chap. 4]. The main idea of this paper is to generate suitable decompositions of functions on $M_j$ by decomposing associated functions on a refined reference configuration $M'_j$. Assuming that...
$M_j'$ is resulting from successive planar refinement of coarse triangles forming a reference configuration $M_0'$, nested finite element spaces on $M_j'$ can be obtained, e.g., by standard nodal interpolation. Existing estimates for this hierarchy provide the desired estimates for an associated hierarchy on $M_j$. In such a way, we are able to derive logarithmic bounds for the convergence rates. The constants solely depend on the ellipticity, the smoothers and on the regularity of the triangles forming the triangular surface $M_j$. Moreover, we obtain exactly the same weights as in the planar case for restriction and prolongation.

The paper is organized as follows. After stating the problem, we introduce the concept of logically nested triangular surfaces, clarifying the connection of $M_j$ and $M_j'$. Section 4 is devoted to a hierarchical decomposition by generalized interpolation. In Sect. 5 we prove logarithmic upper estimates for this hierarchy proving uniformity of convergence for the non-linear case and regularity of the tangential gradient. In Sect. 6 we introduce the concept of logically nested triangular surfaces, clarifying the connection of $M_j$ and $M_j'$ and therefore on $j$. For example, the bilinear form

$$a(v, w) = \int_{M_j} \nabla_{M_j} v(x) \cdot \nabla_{M_j} w(x) \, dx$$

is generated by the Laplace–Beltrami operator. It satisfies (2.2), if the boundary of $M_j'$ is non-empty and homogeneous Dirichlet boundary conditions are prescribed. Moreover, the constants $\alpha, \beta$ are independent of $j$, if $M_j'$ converges to a sufficiently smooth surface $M$ in a suitable way. On these conditions, it is well-known that $u_j$ converges to the solution of the continuous analogue of (2.1) with the same convergence rates as in the planar case [8].

For ease of presentation, we assume from now on that $S_j' = S_j$.

2 Discrete elliptic problems on triangular surfaces

Let $M_j \subset \mathbb{R}^3$ denote a surface consisting of planar triangles $t \in T_j$. To fix the ideas, $M_j$ can be regarded as an approximation of some continuous surface $M$ with $j$ denoting the number of refinement steps. The space $S_j$,

$$S_j = \{ v \in C(M_j) | v_t \in \text{linear \ } \forall t \in T_j \},$$

of linear finite elements on $M_j$ is equipped with the usual Sobolev norms

$$\|v\|^2_{0, M_j} = \int_{M_j} v(x)^2 \, dx, \quad |v|^2_{2, M_j} = \|\nabla_{M_j} v\|^2_{0, M_j},$$

$$\|v\|^2_{1, M_j} = \|v\|^2_{0, M_j} + |v|^2_{1, M_j},$$

which are defined piecewise here, that is, triangle by triangle. The tangential gradient $\nabla_{M_j} v : t \to \mathbb{R}^3$ of a function $v : \mathbb{R}^3 \to \mathbb{R}$ is the tangential part

$$\nabla_{M_j} v = \nabla v - (v \cdot \nabla) v$$

of the gradient of $v$, where $v$ denotes the normal of the actual triangle $t \in T_j$. It depends only on the values of $v$ on $t$ and therefore can be evaluated in an obvious manner for functions defined only on $t$. Correspondingly, the parts of the given norms depend only on the relative position of the vertices of the single triangles to each other and can be evaluated as in the planar case.

We consider the discrete variational problem

$$u_j \in S_j' : \quad a(u_j, v) = \ell(v) \quad \forall v \in S_j', \quad (2.1)$$

Here, $S_j' \subset S_j$ is a subspace of $S_j$, $\ell$ is a linear functional on $S_j'$, and $a(\cdot, \cdot)$ is a symmetric, $S_j'$-elliptic bilinear form. More precisely,

$$\alpha \|v\|^2_{1, M_j} \leq a(v, v) \leq \beta \|v\|^2_{1, M_j} \quad \forall v \in S_j'$$

holds with positive constants $\alpha, \beta$ so that the energy norm $\| \cdot \|$, defined by

$$\|v\|^2 = a(v, v)$$

is equivalent to $\| \cdot \|^2_{1, M_j}$. Usually, both the bilinear form $a(\cdot, \cdot)$ and the right hand side $\ell$ depend on the triangular surface $M_j$ and therefore on $j$. For example, the bilinear form

$$a(v, w) = \int_{M_j} \nabla_{M_j} v(x) \cdot \nabla_{M_j} w(x) \, dx$$

is generated by the Laplace–Beltrami operator. It satisfies (2.2), if the boundary of $M_j$ is non-empty and homogeneous Dirichlet boundary conditions are prescribed. Moreover, the constants $\alpha, \beta$ are independent of $j$, if $M_j$ converges to a sufficiently smooth surface $M$ in a suitable way. On these conditions, it is well-known that $u_j$ converges to the solution of the continuous analogue of (2.1) with the same convergence rates as in the planar case [8].

3 Logically nested triangular surfaces

Let $M_0'$ be a conceptionally coarse, triangular surface consisting of non-degenerate triangles $t' \in T_0'$ and let

$$\phi : M_0' \to M$$

denote a parametrization of a continuous surface $M$ over $M_0'$. We assume that $M_0'$ is conforming in the sense that the intersection of two triangles $t, t' \in T_0'$ is either a common edge, a common vertex or empty. Self-intersections of $M_0'$ are not excluded and $M_0'$ may have a boundary or not. An example is shown in the left picture of Fig. 1. The only condition that we impose on the mapping $\phi$ is later given implicitly, in Definition 2 as follows.

Let $T_0', T_1', \ldots, T_j'$ be a sequence of nested triangulations of $M_0'$ as resulting from standard red/green refinement of $T_0'$ (see, e.g., [2,7], or [12, p. 66]). A triangle $t$ with the vertices $p_i \in \mathbb{R}^3$ is denoted by $t = t(p_1, p_2, p_3)$. For each $k = 0, \ldots, j$, we identify each $t' \in T_k'$ with an associated triangle $t \subset \mathbb{R}^3$ according to

$$T_k' \ni t' = t(p_1', p_2', p_3') \leftrightarrow t = t(p_1, p_2, p_3) \quad (3.1)$$