On Strongly $\pi$-Regular Group Rings

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Abstract. Let $R$ be an associative ring with unit. An element $x \in R$ is said to be left (right) $\pi$-regular if there exist $y \in R$ and a positive integer $n$ such that $x^n = yx^{n+1}$ ($x^n = x^{n+1}y$). If $x$ is both left and right $\pi$-regular, then it is said to be strongly $\pi$-regular. $R$ is said to be a strongly $\pi$-regular ring if all its elements are strongly $\pi$-regular. In this paper we determine some conditions which are necessary or sufficient for a group ring to be strongly $\pi$-regular.

Keywords: strongly $\pi$-regular, regular, group rings

1. Introduction

All rings considered in this paper are associative with unit. An element $x$ in a ring $R$ is said to be left (right) $\pi$-regular if there exist $y \in R$ and a positive integer $n$ such that $x^n = yx^{n+1}$ ($x^n = x^{n+1}y$). If $x$ is both left and right $\pi$-regular, then it is said to be strongly $\pi$-regular. If every element of $R$ is left (right) $\pi$-regular, then $R$ is said to be a left (right) $\pi$-regular ring. The ring $R$ is strongly $\pi$-regular if every element of $R$ is strongly $\pi$-regular. By a result of Dischinger [3], all right $\pi$-regular rings are left $\pi$-regular and vice versa, and all such rings are strongly $\pi$-regular. If given any $x \in R$ there exists $y \in R$ such that $xyx = x$, then $R$ is said to be a von Neumann regular ring. In what follows, we shall refer to von Neumann regular rings as just regular rings.

It is known that a strongly $\pi$-regular ring is not necessarily regular and vice versa. For example, the ring of $n \times n$ lower triangular matrices over a field $\mathbb{F}$ is strongly $\pi$-regular but not regular while the ring of endomorphisms $\text{End}_D(V)$, where $V$ is an infinite dimensional vector space over the division ring $D$ is regular but not strongly $\pi$-regular. More properties of strongly $\pi$-regular and regular rings can be found for example in [1] and [4].

Necessary and sufficient conditions for a group ring to be regular have been known since the late fifties and early sixties (see [5, Theorem 3.15] for example). In this paper we study strongly $\pi$-regular group rings and obtain some conditions which are necessary or sufficient for a group ring to be strongly $\pi$-regular.
2. Some Preliminaries

Let \( R \) be a ring and suppose that \( R \) is not right \( \pi \)-regular. Then there exists an element \( x \in R \) such that given any positive integer \( n, x^n \neq x^{n+1}y \) for any \( y \in R \). We then have a descending chain

\[
xR \supseteq x^2R \supseteq x^nR \supseteq x^{n+1}R \supseteq \cdots
\]

of right ideals of \( R \) which does not terminate; thus \( R \) is not artinian. It follows from this that artinian rings must be strongly \( \pi \)-regular.

It is straightforward to show that homomorphic images of strongly \( \pi \)-regular rings are strongly \( \pi \)-regular.

Let \( R \) be a ring and \( G \) a group. We shall denote the group ring of \( G \) over \( R \) as \( RG \). For any element \( r = \sum_{g \in G} r_g g \in RG \), the support of \( r \), written as \( \text{Supp}(r) \), is the subset of \( G \) consisting of all those \( g \in G \) such that \( r_g \neq 0 \). Since \( r_g \neq 0 \) for only finitely many \( g \in G \), \( \text{Supp}(r) \) is a finite subset of \( G \). The augmentation ideal of \( RG \) is the ideal of \( RG \) generated by \( \{1 - g | g \in G\} \). We shall use \( \Delta \) to denote the augmentation ideal of \( RG \). It is known (see [5] for example) that \( R \) is a homomorphic image of \( RG \) since \( RG/\Delta \cong R \).

3. Strongly \( \pi \)-Regular Group Rings

The main result in this section is as follows:

**Theorem 3.1.** Let \( R \) be a ring and \( G \) a group. If \( (R/P)G \) is strongly \( \pi \)-regular for every prime ideal \( P \) of \( R \), then \( RG \) is strongly \( \pi \)-regular.

**Proof.** Suppose to the contrary that \( RG \) is not strongly \( \pi \)-regular. Then there exists an element \( x \in RG \) such that for any positive integer \( n, x^n \neq x^{n+1}y \) for any \( y \in RG \). Therefore the sequence \( xRG \supseteq x^2RG \supseteq \cdots \supseteq x^nRG \supseteq x^{n+1}RG \supseteq \cdots \) of right ideals of \( RG \) does not terminate. Let \( F \) be the set of all ideals \( I \) of \( R \) such that the sequence \( (x + IG)(RG/IG) \supseteq (x + IG)^2(RG/IG) \supseteq \cdots \) does not terminate. Note that \( F \neq \emptyset \) since \( (0) \in F \). Furthermore, \( F \) is partially ordered by inclusion. Let \( (I_z)_{z \in \Omega} \) be a chain of elements of \( F \) and let \( J = \bigcup_{z \in \Omega} I_z \). Clearly, \( J \) is an ideal of \( R \) and \( I_z \subseteq J \) for all \( z \in \Omega \). We show that \( J \in F \). Suppose that \( J \notin F \). Then \( z = x^n - x^{n+1}r \in JG \) for some \( r \in RG \) and some positive integer \( n \). Since \( \text{Supp}(z) \) is finite, there exists some \( z \in \Omega \) such that \( z \in I_zG \). It follows that the sequence

\[
(x + I_zG)(RG/I_zG) \supseteq (x + I_zG)^2(RG/I_zG) \supseteq \cdots
\]

terminates, which is a contradiction. Therefore \( J \in F \) and thus by Zorn's Lemma, \( F \) contains a maximal element \( M \). Since \( (R/M)G \supseteq RG/MG \) is not strongly \( \pi\)-