Necessary and Sufficient Conditions for Oscillation of Delay Parabolic Differential Equations

Wei Nian Li\textsuperscript{a,b}

\textsuperscript{a} Department of Mathematics, Qufu Normal University, Shandong 273165, China
E-mail: wnli@xinhuanet.com

Bao Tong Cui\textsuperscript{b}

\textsuperscript{b} Department of Mathematics, Binzhou Normal College, Shandong 256604, China

AMS Subject Classification (2000): 35B05, 35R10

Abstract. Necessary and sufficient conditions for oscillation of delay parabolic differential equations are obtained.

Keywords: Oscillation; Parabolic differential equation; Delay

1. Introduction

In the past few years, Mishev and Bainov [4], Yu and Chen [5] have studied the necessary and sufficient conditions for oscillation of neutral partial differential equations with constant parameters. Recently, Cui and Li [1] established a necessary and sufficient condition for oscillation of parabolic equations of the form

\[
\frac{\partial}{\partial t} u(x, t) = a(t) \Delta u(x, t) + \sum_{k=1}^{s} a_k(t) \Delta u(x, t - \rho_k(t)) - \sum_{j=1}^{m} q_j(t) u(x, t - \sigma_j(t)),
\]

\[(x, t) \in \Omega \times [0, \infty) \equiv G,\]

with the boundary condition

\[
u(x, t) = 0, \quad (x, t) \in \partial \Omega \times [0, \infty),\]

where \( \Omega \) is a bounded domain in \( R^N \) with a piecewise smooth boundary \( \partial \Omega \), and \( \Delta \) is the Laplacian in the Euclidean \( N \)-space \( R^N \); \( a, a_k, q_j \in C([0, \infty); [0, \infty)), \rho_k, \sigma_j \in C([0, \infty); [0, \infty)); \lim_{t \to \infty} (t - \rho_k(t)) = \lim_{t \to \infty} (t - \sigma_j(t)) = \infty, \quad k \in I_s = \{1, 2, \ldots, s\}, j \in I_m = \{1, 2, \ldots, m\} \).

The main result in [1] is as follows:

\footnote{This work is supported by the Natural Science Foundation of Shandong Province, China (Y2001A03).}
Theorem A. Suppose that there exists a nonempty set \( I^* \subseteq I_{s+m} = \{1, 2, \ldots, s + m\} \) such that

(B1) \( T_i(t) > 0 \) for \( t \geq 0, i \in I^*; \)
(B2) \( \sum_{i \in I^*} P_i(t) > 0 \) for \( t \geq 0, \)

where

\[
T_i(t) = \begin{cases} 
  \rho_i(t), & 1 \leq i \leq s, \\
  \sigma_{i-s}(t), & s + 1 \leq i \leq s + m,
\end{cases}
\]

\[
P_i(t) = \begin{cases} 
  a_i(t), & 1 \leq i \leq s, \\
  q_{i-s}(t), & s + 1 \leq i \leq s + m.
\end{cases}
\]

Then every solution of the problem (1), (2) is oscillatory in \( G \) if and only if the differential inequality

\[
V'(t) + \sum_{k=1}^{s} a_k(t) V(t - \rho_k(t)) + \sum_{j=1}^{m} q_j(t) V(t - \sigma_j(t)) \leq 0 \quad (3)
\]

has no eventually positive solutions.

In this paper, we consider the boundary conditions (2) and

\[
\frac{\partial u(x,t)}{\partial N} = 0, \quad (x,t) \in \partial \Omega \times [0, \infty), \quad (4)
\]

where \( N \) is the unit exterior normal vector to \( \partial \Omega \). Some necessary and sufficient conditions for oscillation of the problem (1), (2) and (1), (4) are obtained.

The function \( u \in C^2(G) \cap C^1(\partial G) \) is said to be a solution of the problem (1), (2) (or (1), (4)) if it satisfies (1) in the domain \( G \) and the boundary condition (2) (or (4)). The solution \( u(x,t) \) of the problem (1), (2) (or (1), (4)) is said to be oscillatory in the domain \( G = \Omega \times [0, \infty) \) if for any positive number \( \mu \) there exists a point \( (x_0, t_0) \in \Omega \times [\mu, \infty) \) such that the equality \( u(x_0, t_0) = 0 \) holds.

2. Main Results

The following facts are useful in the proof of our main results.

Lemma [2]. Consider the equation

\[
V'(t) + \sum_{i=1}^{n} p_i(t) V(t - \sigma_i(t)) = 0. \quad (5)
\]

Suppose that the following conditions hold: