Necessary and Sufficient Conditions for Oscillation of Delay Parabolic Differential Equations

Wei Nian Li\textsuperscript{a,b}
\textsuperscript{a}Department of Mathematics, Qufu Normal University, Shandong 273165, China
E-mail: wnli@xinhuanet.com

Bao Tong Cui\textsuperscript{b}
\textsuperscript{b}Department of Mathematics, Binzhou Normal College, Shandong 256604, China

AMS Subject Classification (2000): 35B05, 35R10

Abstract. Necessary and sufficient conditions for oscillation of delay parabolic differential equations are obtained.

Keywords: Oscillation; Parabolic differential equation; Delay

1. Introduction

In the past few years, Mishev and Bainov [4], Yu and Chen [5] have studied the necessary and sufficient conditions for oscillation of neutral partial differential equations with constant parameters. Recently, Cui and Li [1] established a necessary and sufficient condition for oscillation of parabolic equations of the form

\begin{equation}
\frac{\partial}{\partial t} u(x, t) = a(t) \Delta u(x, t) + \sum_{k=1}^{s} a_k(t) \Delta u(x, t - \rho_k(t)) - \sum_{j=1}^{m} q_j(t) u(x, t - \sigma_j(t)),
\end{equation}

\begin{equation}
(x, t) \in \Omega \times [0, \infty) \equiv G,
\end{equation}

with the boundary condition

\begin{equation}
\frac{\partial}{\partial n} u(x, t) = 0, \quad (x, t) \in \partial\Omega \times [0, \infty),
\end{equation}

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ with a piecewise smooth boundary $\partial\Omega$, and $\Delta$ is the Laplacian in the Euclidean $N$-space $\mathbb{R}^N$; $a, a_k, q_j \in C([0, \infty); [0, \infty))$, $\rho_k, \sigma_j \in C([0, \infty); [0, \infty))$, $\lim_{t \to \infty} (t - \rho_k(t)) = \lim_{t \to \infty} (t - \sigma_j(t)) = \infty$, $k \in I_s = \{1, 2, \ldots, s\}$, $j \in I_m = \{1, 2, \ldots, m\}$.

The main result in [1] is as follows:

\textsuperscript{1}This work is supported by the Natural Science Foundation of Shandong Province, China (Y2001A03).
Theorem A. Suppose that there exists a nonempty set $I^* \subseteq I_{s+m} = \{1, 2, \ldots, s + m\}$ such that

(B1) $T_i(t) > 0$ for $t \geq 0, i \in I^*$;
(B2) $\sum_{i \in I^*} P_i(t) > 0$ for $t \geq 0$, where

$$T_i(t) = \begin{cases} 
\rho_i(t), & 1 \leq i \leq s, \\
\sigma_{i-s}(t), & s + 1 \leq i \leq s + m,
\end{cases}$$

$$P_i(t) = \begin{cases} 
\alpha_i(t), & 1 \leq i \leq s, \\
\sigma_{i-s}(t), & s + 1 \leq i \leq s + m.
\end{cases}$$

Then every solution of the problem (1), (2) is oscillatory in $G$ if and only if the differential inequality

$$V'(t) + \sum_{i=1}^{s} a_i(t) V(t - \rho_i(t)) + \sum_{j=1}^{m} q_j(t) V(t - \sigma_j(t)) \leq 0$$

has no eventually positive solutions.

In this paper, we consider the boundary conditions (2) and

$$\frac{\partial u(x, t)}{\partial N} = 0, \quad (x, t) \in \partial \Omega \times [0, \infty),$$

where $N$ is the unit exterior normal vector to $\partial \Omega$. Some necessary and sufficient conditions for oscillation of the problem (1), (2) and (1), (4) are obtained.

The function $u \in C^2(G) \cap C^1(\overline{G})$ is said to be a solution of the problem (1), (2) (or (1), (4)) if it satisfies (1) in the domain $G$ and the boundary condition (2) (or (4)). The solution $u(x, t)$ of the problem (1), (2) (or (1), (4)) is said to be oscillatory in the domain $G = \Omega \times [0, \infty)$ if for any positive number $\mu$ there exists a point $(x_0, t_0) \in \Omega \times [\mu, \infty)$ such that the equality $u(x_0, t_0) = 0$ holds.

2. Main Results

The following facts are useful in the proof of our main results.

Lemma [2]. Consider the equation

$$V'(t) + \sum_{i=1}^{n} p_i(t) V(t - \sigma_i(t)) = 0.$$  \hspace{1cm} (5)

Suppose that the following conditions hold: