**Abstract** In numerically simulated vibrated beds of powder, we measure temperature under convection by the generalized Einstein’s relation. The spatial temperature distribution turns out to be quite uniform except for the boundary layers. In addition to this, temperature remains uniform even if segregation occurs. This suggests the possibility that there exists some *thermal equilibrium state* even in a vibrated bed of powder. This finding may lead to a unified view of the dynamic steady state of granular matter.

**Keywords** Einstein’s relation · temperature · vibrated bed of powder · numerical simulation · segregation

**1 Introduction**

Granular matter is characterized by a set of macroscopic particles, e.g., sand, snow fall, sugar, salt and rice. Recently many physicists became interested in the dynamics of granular matter [1], because it shows many striking features. These include convection, segregation, surface waves, and flows. They are hard to understand by the word of physics, although it is relatively easy to reproduce them numerically.

For example, the concept of granular temperature [1] is often used to describe the dynamical state of powder. However, in contrast to rapid granular flow [2], the granular temperature turns out not to be a good measure for dense granular matter. For example, granular temperature in vibrated beds of powder is anisotropic [3], which is not a favorable feature.

On the other hand, Wildman and Huntley [4] have found useful a temperature that is given by the mean squared displacement curves for short times. Although it sounds promising, their definition is valid only when convective flow is negligible.

Recently Makse and Kurchan [5] have numerically found that the temperature defined by Einstein’s relation is well defined and is independent of the size of the particles in granular matter. In contrast to that of Wildman and Huntley [4], Makse and Kurchan’s definition is valid even if there is a non-negligible flow. Also D’Anna et al [6] have found that their variant using AC frequency is also well defined. Thus we may expect that transport phenomena of granular particles can give us a more suitable definition of temperature than the conventional granular temperature.

In spite of these successful applications of Einstein’s relation to define the temperature in the granular matter, its application is very limited. This is because the conventional Einstein’s relation can be defined only in a spatially uniform system. In order to apply Einstein’s relation, we need to understand the long time behavior of tagged particles. If the system is not uniform at all, the tagged particle will travel around various places having various temperatures. Thus what we can measure is a spatially averaged temperature, which cannot reflect spatial structure in the system. On the other hand, if the system is uniform, temperature is uniform, too. This means that we can never know if the thermal equilibrium really stands or not from the measurement of temperature by Einstein’s relation. However, as demonstrated by Wildman and Huntley [4], it is not always necessary to observe long time behavior in order to derive temperature from the transport phenomena.

In this paper, we have developed a heuristic method to measure temperature by Einstein’s relation in the strongly non-uniform system. In §2, following Makse and Kurchan [5], we introduce the temperature using Einstein’s relation. In §3, the heuristic approach to define local temperature is presented, and in §4 this definition is further generalized so as to measure temperature in the system under the flow. The numerical results in vibrated beds are shown in §5, and how segregation affects the temperature is investigated in §6. In §7, we have discussed about the validity of our results; §8 contains summary and conclusion.
2 The Einstein’s relation and temperature

Suppose there is a particle which obeys a random walk. So as to be simple, we restrict its motion to the one dimensional space. If \( x(t) \) is the coordinate of the particle at time \( t \), then the diffusion constant \( D \) is defined as

\[
\langle (x(t) - x(0))^2 \rangle = 2Dt,
\]
where \( \langle \cdots \rangle \) means the ensemble average. On the other hand, the mobility \( \mu \) can be defined as

\[
\langle x(t) - x(0) \rangle = \mu ft,
\]
where \( f \) is the applied force. When combining the above two equations with Einstein’s relation

\[
T \equiv \frac{D}{\mu},
\]
where \( T \) is temperature, we get

\[
T = \frac{f}{2} \langle (x(t) - x(0))^2 \rangle.
\]

Recently, Makse and Kurchan [5] used this equation as the definition of temperature in granular matter. They have performed three dimensional distinct element method (DEM) for the periodic sheared system consisting of a binary set of both large and small particles. They measured \( D \) and \( \mu \) along the direction perpendicular to the shear and found that the measured temperature is the same for both large and small particles.

More recently, D’Anna et al [6] have experimentally found that the fluctuation-dissipation ratio

\[
\frac{S(\omega)\omega}{4\chi''(\omega)}
\]
can be a well defined temperature for the vibrated bed, where \( S(\omega) \) is the noise power spectrum density of the angular frequency \( \omega \) and \( \chi'' \) is the imaginary part of the complex susceptibility. Since this can be regarded as the generalization of the temperature introduced by Makse and Kurchan to the frequency dependent one, these kinds of definition of temperature seem to be valid for granular matter.

However, there is a difficulty to apply the above definitions to general cases. This is because it is impossible to measure the spatial dependence of temperature using these definitions. Makse and Kurchan’s definition is valid only for the long time limit, and D’Anna et al’s procedure is used to measure temperature only when we can regard a whole vibrated bed as a thermal bath. Thus, it is rather difficult to relate temperature to the statistical mechanics, because in the statistical mechanics local temperatures must be defined. The definitions above cannot provide such information.

First, we subdivide the whole system into smaller cells. Each cell has an index \( i \). Next, we define the transition probability of a particle from the \( i \)th cell to the \( j \)th cell \( P(i, j) \). Suppose the spacing between cells is \( \ell \), a random walker must travel the distance of \( \ell \) in order to go from cell \( i \) to cell \( i+1 \). Thus the averaged time \( t_0 \) until this arises is obtained as

\[
t_0 = \ell^2/2D.
\]

The escape probability of a particle from cell \( i \) is

\[
P_{\text{out}}(i) \equiv \sum_{k=\pm 1} P(i+k, i; f=0) \propto 1/t_0 = 2D/\ell^2.
\]

On the other hand, when a drift force \( f \) exists,

\[
t_0 = \ell/f \mu,
\]
then the flow \( J(i) \) along the \( f \) direction is

\[
J(i) \equiv \ell \sum_{k=\pm 1} k P(i+k, i; f \neq 0) \propto \ell/t_0 = f \mu.
\]

Thus

\[
T_i = \frac{D}{\mu} = \frac{P_{\text{out}}(i)\ell^2/2}{J(i)/f} = \frac{f}{2} \frac{P_{\text{out}}(i)\ell^2}{J(i)}
\]
is the local temperature at cell \( i \). Of course we do not insist that this is rigorous, but only regard it as a heuristic argument. Also, in order to remove the dependency on \( f \) we employ the following definition

\[
T_i^{\text{eff}} \equiv \lim_{f \to 0} T_i
\]
of the effective temperature.

4 The definition of effective temperature in the system with flow

As an example to test the above heuristic definition of the effective temperature, we employ a numerical simulation of a vibrated bed of powder. In the vibrated bed of powder considered here, there are convective motions of particles. Thus \( \langle x(t) - x(0) \rangle \) is not zero even if \( f \) is equal to 0. Furthermore, the vibrated bed we deal with is a two dimensional system. Therefore we need some additional modification of the previous definition of the effective temperature \( T_i^{\text{eff}} \).

First we subdivide the system into \( N_1 \times N_2 \) cells, where \( N_1 \ell \) is the horizontal size of system and \( N_2 \ell \) is the vertical size of the system. Then we denote the cell as \( (i, j) \) if the cell is the \( i \)th in horizontal direction and the \( j \)th in vertical direction. The flow vector at cell \( (i, j) \) is defined as

\[
J(i, j; f = 0) \equiv \ell \sum_{k=\pm 1} k \left( P(i+k, i; j; f = 0) P(i; j+k, j; f = 0) \right),
\]
where \( P(i', i; j) \) is the transition probability from \( (i, j) \) to \( (i', j) \) and \( P(i; j', j) \) is the transition probability from \( (i, j) \) to \( (i, j') \) (See Fig. 1).