Multidimensional Polyhedral Gamut Representation of Reflective Objects
and Calculation of Metamer Ensembles

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The spectral reflectance of most reflective objects, such as natural objects and color hardcopy, is relatively smooth and can be approximated by several principal components with high accuracy. Although the subspace spanned by those principal components represents a space in which reflective objects exist, it does not provide the limit of the object distribution or the gamut. In this paper we propose to represent the gamut of reflective objects as a convex polyhedron in the subspace spanned by several principal components. The concept of the polyhedral gamut representation based on the spectral reflectance database and its application to the calculation of metamer ensembles are described. The color-mismatch volume caused by different illuminant and/or observer for a metamer ensemble is also calculated and compared with the theoretical ones.

Key words: color gamut, spectral reflectance, metamerism, convex polyhedron, color mismatch volume, spectral reflectance database

1. Introduction

It is known that the spectral reflectance of most reflective objects, such as natural objects or color hardcopy, is relatively smooth and can be approximated by several base spectral functions with high accuracy. One set of such functions can be found by principal component analysis (PCA) for a set of samples collected appropriately. The subspace spanned by the principal components (PCs) obtained represents a space in which the spectral reflectance of reflective objects exists. However, it still has infinite extension, namely, it does not specify the gamut or the limit in which the samples are distributed. In this paper, we propose to represent a spectral gamut of objects by a convex polyhedron in this subspace. If spectral reflectance is approximated by \(N\) principal components, it can be represented by \(N\) coefficients for the corresponding PCs. A polyhedron representing the gamut is formed by an outer surface. The outer surface is composed of many patches that we call facets, each of which has \(N\) apexes or \(N\) samples. Thus, defining a gamut is equivalent to listing the facets of the polyhedron.

If such a practical gamut of reflective objects in the spectral domain becomes available, it can be used for the calculation of metamer ensembles. Papers on finding metamer ensembles have been published. Those studies were on theoretical ensembles subject to the physical constraint that the upper limit of reflectance be 1 and the lower limit 0, however, this constraint is too loose for practical use. Stiles et al. added the constraint of frequency-limitation for spectral reflectance functions to the physical constraint and recalculated metamer ensembles. Although this technique provides a more practical metamer, the selection of the base spectral functions and the range of coefficients for those functions are not based on practical object statistics. Drew and Funt used principal components to find the best metamer for color reproduction but did not provide a metamer ensemble.

A metamer calculated using a principal-component-based polyhedral gamut has a distinct limit and therefore is more practical. When sensor responses are given, the mathematical metamer ensemble is expressed by a subspace in the spectral reflectance domain. Once the object gamut is specified, the metamer ensemble is given by a cross section of the polyhedral gamut and the subspace determined by the sensor responses. In this paper, metamer ensembles are calculated for some given illuminant and observer conditions, then the extent of the ensemble viewed under a different illuminant or by a different observer in a color space is calculated. Those results are compared with theoretical ones presented by Schmitt and Stiles et al.

Actually, gamut representation using several PCs and its application have already been presented by Finlayson and Morovic. They utilized the several PCs to determine the range of metamers using the maximum and minimum values of the PC coefficients of the samples and successfully applied the results to color correction. They also used the same idea to evaluate the performance of the Sharp Adaptation Transform that they have developed for color transform including adaptation. We have been conducting the present study independently of Finlayson and Morovic and found that the basic idea is similar in the sense that one approximates the spectral reflectance by several PCs and represents a gamut on the basis of the sample distribution. However, there is still a distinct difference in determination of the gamut. As seen later, we treat a gamut as a convex polyhedron that just covers all samples. In this case, the apexes of the convex hull are actual samples. On the other hand, Finlayson and Morovic’s convex polyhedron is a simpler one in which the limits of each axis are given by the maximum and minimum values along the axis among the samples. Detailed comparison is described in the last section.

2. Method

2.1 Polyhedral representation of object gamut

In this paper, bold lowercase letters and bold capital letters represent column vectors and matrices, respectively, while plain letters represent scalar values. We assume that
all spectral data are sampled into \( P \) elements over a visible range. For example, the spectral reflectance of an object is represented by a column vector,
\[
f = [f_1 \cdots f_P]^T,
\]
where \([\cdot]^T\) denotes the transposition.

Now suppose that a database with a sufficient number of samples is given. In this study, as the set of samples for performing PCA, we use original samples plus those with a negative sign, as did Maloney.\(^5\) With this treatment, an approximated spectral reflectance can be expressed by a linear combination of PCs without adding the mean spectral reflectance and data handling becomes easy.

We denote PC vectors by column vectors, \( k_i, \) \( i = 1, \ldots, N. \) Here, \( N \) is the number of PCs used for approximation and the order of the subscript corresponds to a descending order of variance. The original spectral reflectance can be expressed using these PC vectors as
\[
f \approx \sum_{i=1}^{N} a_i k_i = Ka
\]
where
\[
a = [a_1 \cdots a_N]^T
\]
\[
K = [k_1 \cdots k_N].
\]
The coefficient vector \( a \) is calculated from \( f \) by
\[
a = K^T f
\]
Expressing the gamut by the PC coefficients that can approximate the original spectral reflectance is approximately equivalent to expressing the gamut in the spectral reflectance domain. Thus we shall represent a reflective object by a PC coefficient vector \( a. \) The samples in the database are actually measured ones and therefore physically possible. We add the ideal black object, \( f = 0 \) or \( a = 0, \) to the database since it is physically possible. Next, let us consider the following linear combination of \( n \) arbitrary but linear-independent samples selected from the database,
\[
a = \sum_{i=1}^{n} w_i a_i, \quad 0 \leq w_i, \quad \sum_{i=1}^{n} w_i \leq 1.
\]
Here, the value \( n \) itself is also an arbitrary natural number less than or equal to \( N \) and the suffix \( i \) denotes \( i \)th sample. \( \{w_i\} \) represents the weights in the linear combination. From the condition on the weights, this linear combination yields a vector inside a polyhedron formed by the samples, \( a_i, \) \( i = 1, \ldots, n \) and \( a = 0. \) This linear combination can also be treated as a possible color because such a combination corresponds to an area-weighted mixture of the samples in a measurement aperture and is physically possible. In this paper, therefore, we determine the outer surface of the database by a convex polyhedron and regard it as the object gamut. All possible combination of samples lie inside the convex polyhedron or convex hull.

It should be noted that the above-defined gamut is different from a commonly defined gamut. Considering a color gamut of a printer, for example, the gamut means a volume in a color space that the printed patches themselves form and area-weighted color mixtures of patches are not included in this volume. Such defined volume is not necessarily convex. When treating metamerism, however, we consider that the above mentioned color mixture should be included in a gamut.

To clarify the properties of the convex polyhedron, we consider the cases that \( N = 2 \) (2D) and \( N = 3 \) (3D) as shown in Fig. 1, for simplicity. In the 2D case, the outer surface of the polyhedron is formed by line segments with two apexes each. In the 3D case, the outer surface of the polyhedron is formed by triangular patches with three apexes each. For consistent expression over all dimensions, we call an element forming the outer surface of the polyhedron a facet. In general, the outer surface of a hyper-polyhedron in \( N \)-dimensional space is formed by facets with \( N \) apexes each. We define a facet by a set of \( N \) apexes and represent the \( i \)th facet as
\[
S_{\text{facet}} = \{a^{(i,j)} | j = 1, \ldots, N\}.
\]
Here, \( a^{(i,j)} \) represents the \( j \)th apex of the \( i \)th facet.

Once a set of samples is given, facets of the convex polyhedron can be found by a proper algorithm. Some open software such as “qhull” are available to find a convex

![Fig. 1. Schematic illustration of a convex polyhedron representing an object gamut. (a) Two-dimensional case, (b) three-dimensional case.](image-url)