Steady-states and kinetics of ordering in bus-route models: connection with the Nagel-Schreckenberg model

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Abstract. A Bus Route Model (BRM) can be defined on a one-dimensional lattice, where buses are represented by “particles” that are driven forward from one site to the next with each site representing a bus stop. We replace the random sequential updating rules in an earlier BRM by parallel updating rules. In order to elucidate the connection between the BRM with parallel updating (BRMPU) and the Nagel-Schreckenberg (NaSch) model, we propose two alternative extensions of the NaSch model with space/time-dependent hopping rates. Approximating the BRMPU as a generalization of the NaSch model, we calculate analytically the steady-state distribution of the time headways (TH) which are defined as the time intervals between the departures (or arrivals) of two successive particles (i.e., buses) recorded by a detector placed at a fixed site (i.e., bus stop) on the model route. We compare these TH distributions with the corresponding results of our computer simulations of the BRMPU, as well as with the data from the simulation of the two extended NaSch models. We also investigate interesting kinetic properties exhibited by the BRMPU during its time evolution from random initial states towards its steady-states.

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1 Introduction

Systems of interacting particles driven far from equilibrium are of current interest in statistical physics \cite{1-5}. Microscopic models of such systems often capture some aspects of vehicular traffic. In such “particle-hopping” models of vehicular traffic the particles represent vehicles and the nature of the interactions among these particles is determined by the manner in which the vehicles influence the motion of each other \cite{6-8}. The dynamics of these models are often formulated in terms of “update rules” using the language of cellular automata (CA) \cite{9}. For example, the Nagel-Schreckenberg (NaSch) model \cite{10,11} is the most popular minimal CA model of vehicular traffic on highways while, to our knowledge, the first CA model of city traffic was developed by Biham, Middleton and Levin \cite{12}. The results obtained for these models, using the techniques of statistical mechanics, are not only of fundamental interest for understanding truly nonequilibrium phenomena but may also find practical use in traffic science and engineering, \cite{13-15}. Among such results are the time-headway and distance-headway distributions. The time-headway (TH) is defined as the time interval between the departures (or arrivals) of two successive vehicles recorded by a detector placed at a fixed position on the route while the distance between the successive vehicles can be defined as the corresponding distance-headway (DH). The distributions of TH and DH not only contain detailed informations on the nature of the spatio-temporal organization of the vehicles but are also of practical interest to traffic engineers because larger headways provide greater margins of safety whereas higher capacities of the highway require smaller headways.

In a 1998 paper, O’Loan et al. \cite{16} have developed a one-dimensional lattice model of bus-route where the buses are represented by particles which move from one site to the next; each site of this model represents a bus stop along the route. The motion of the buses in this bus route model with random sequential updating (BRMRSU) is strongly influenced by the passengers waiting at the bus stops. The BRMRSU model may be viewed as a generalization of a simple particle-hopping model, namely, the totally asymmetric simple exclusion process (TASEP) by coupling the dynamics of the particles to another new variable which represents the presence (or absence) of passengers waiting at the bus stops. The bus route model in \cite{16} does not deal with overcrowded buses; it implicitly assumes that either the buses have infinite capacity or that the passenger arrival rate is slow enough to avoid overcrowding.
The BRMRSU exhibits a Bose-Einstein-condensation-like phenomenon which has been observed earlier in the TASEP and in the NaSch model when quenched random hopping rates are associated with the particles [17–19]. However, unlike the stable Bose-Einstein-condensed states observed at sufficiently low densities in the TASEP (and in the NaSch model) with random hopping rates, those in the BRMRSU are metastable. The main characteristic of the spatially-inhomogeneous Bose-Einstein-condensed state is the existence of a macroscopically long gap in front of a cluster of vehicles led by the slowest one. In finite systems, for small \( \lambda \) (\( \lambda \) is the rate of passenger arrival at a bus stop), the bus clusters (or, equivalently, the gaps between clusters) in the BRMRSU exhibit interesting coarsening phenomena as the system evolves from a random initial state. O’Loan et al. [16] find that, after sufficiently long time, the typical size of the large gaps in the system grows with time \( t \) according to a power growth law \( t^{1/2} \).

In this paper we use a BRM with parallel updating (BRMPU) which is obtained from the BRMRSU [16] by replacing the random sequential updating rule with parallel updating, with the aim of relating it with the NaSch model where updating is done in parallel. We also propose here two extensions of the NaSch model (from now onwards referred to as the models Y and Z) by replacing the constant hopping rates with two different time-/space-dependent hopping rates which we shall specify explicitly in Section 2. We have computed the TH distributions in the steady-states of the BRMPU as well as in the models Y and Z through computer simulations. Comparison of these distributions is shown in section III. Such comparisons elucidate the connection between the BRMPU and the NaSch model. Then, approximating the BRMPU as a generalization of the NaSch model with a time-dependent hopping rate for the buses, we calculate in Section 4 the TH distribution in the BRMPU from the corresponding analytical expression in the NaSch model. We compare the TH distributions thus derived from analytical considerations with the corresponding results of computer simulations of the BRMPU. These comparisons do not merely point out the regimes of validity of our analytical results but also indicate the differences arising from the different natures of the low-density steady-states of the BRMRSU and the NaSch model. Finally, we investigate in Section 5, interesting kinetic phenomena at low densities of the BRMRSU by computing the appropriate correlation functions (to be defined in Sect. 5). We extract the universal laws governing the growth of the clusters of buses in finite samples of BRMPU at low densities where the system approaches a Bose-Einstein-like “condensed” state evolving from random initial states.

2 The models and methods

Let us first summarize how the totally asymmetric exclusion process (TASEP) [1–3], the NaSch model [10] and the bus route models [16] are defined.

2.1 TASEP and the NaSch model

In the “particle-hopping” models of traffic the position, speed, acceleration as well as time are treated as discrete variables. In this approach, a lane is represented by a one-dimensional lattice. Each lattice site represents a “cell” which can be either empty or occupied by at most one “vehicle” at a given instant of time. At each discrete time step \( t \to t + 1 \), the state of the system is updated following a well-defined prescription. In the TASEP a randomly chosen particle can move forward, by one lattice spacing, with probability \( q \) if the lattice site immediately in front of it is empty. In the NaSch model, the speed \( v \) of each vehicle can take one of the \( v_{\text{max}} + 1 \) allowed integer values \( v = 0, 1, ... , v_{\text{max}} \). If the random-sequential updating scheme of the TASEP is replaced by parallel updating then it becomes identical to the NaSch model with \( v_{\text{max}} = 1 \) and random braking probability \( p = 1 - q \). Our interest in the NaSch model is to unravel its connections to the TASEP. For this purpose, we only need the NaSch model with \( v_{\text{max}} = 1 \). Thus in what follows, by the NaSch model, we shall mean NaSch model with \( v_{\text{max}} = 1 \), unless explicitly stated otherwise.

2.2 BRM with parallel and random-sequential updatings

In the BRM [16] each of the lattice sites represents a bus stop and these stops are labeled by an index \( i \) (\( i = 1, 2, ..., L \)) [16]. In each step of updating, each bus attempts to hop from one stop to the next. Note that in the TASEP and the NaSch model one can label the lattice sites by the index \( i \) (\( i = 1, 2, ..., L \)) and describe the state of each of the sites by associating a variable \( \sigma_i \) with it; \( \sigma_i = 1 \) if the site \( i \) is occupied and \( \sigma_i = 0 \) if the site \( i \) is empty. In contrast, in the BRM, the two binary variables \( \sigma_i \) and \( \phi_i \) are assigned to each site \( i \): (i) If the site \( i \) is occupied by a bus then \( \sigma_i = 1 \); otherwise \( \sigma_i = 0 \). (ii) If the site \( i \) has passengers waiting for a bus then \( \phi_i = 1 \); otherwise \( \phi_i = 0 \). A site cannot have both \( \sigma_i = 1 \) and \( \phi_i = 1 \) simultaneously since a site cannot have simultaneously a bus and waiting passengers. The state of the system is updated according to the following random sequential update rules: a site \( i \) is picked up at random. Then, (i) if \( \sigma_i = 0 \) and \( \phi_i = 0 \) (i.e., site \( i \) contains neither a bus nor waiting passengers), then \( \phi \to 1 \) with probability \( \lambda \), where \( \lambda \) is the probability per unit time of the arrival (i.e., the arrival rate) of the first passenger at the empty bus stop. (Arrival of the subsequent passengers does not affect the time evolution.) (ii) If \( \sigma_i = 1 \) (i.e., there is a bus at the site \( i \)) and \( \sigma_{i+1} = 0 \), then the hopping rate \( \mu \) of the bus from site \( i \) to \( i + 1 \) is defined as follows: (a) if \( \phi_{i+1} = 0 \), then \( \mu = \alpha \) but (b) if \( \phi_{i+1} = 1 \), then \( \mu = \beta \), where \( \alpha \) is the hopping rate of a bus onto a stop which has no waiting passengers and \( \beta \) is the hopping rate onto a stop with waiting passenger(s). Generally, \( \beta < \alpha \), which reflects the fact that a bus has to slow down when it has to pick up passengers. In the BRMRSU one can set \( \alpha = 1 \) without loss of generality. However, for reasons which will become clear soon, we shall keep \( \beta < \alpha < 1 \). When a bus