A path integral approach to inclusive processes

O. Nachtmann, A. Rauscher
Institut für Theoretische Physik, Universität Heidelberg, Philosophenweg 16, 69120 Heidelberg, Germany

Received: 8 March 2000 / Published online: 6 July 2000 – © Springer-Verlag 2000

Dedicated to Prof. Franz Wegner on the Occasion of his 60th Birthday

Abstract. The single-particle inclusive differential cross-section for a reaction \(a + b \to c + X\) is written as the imaginary part of a correlation function in a forward scattering amplitude for \(a + b \to a + b\) in a modified effective theory. In this modified theory the interaction Hamiltonian \(H_I\) equals \(H_I\) in the original theory up to a certain time. Then there is a sign change and \(H_I\) becomes nonlocal. This is worked out in detail for scalar field models and for QED plus the abelian gluon model. A suitable path integral for direct calculations of inclusive cross sections is presented.

1 Introduction

In this article we consider inclusive cross sections, i.e. reactions of the type
\[
a(p_1) + b(p_2) \rightarrow c(p_3) + X, \tag{1.1}\]
where \(a, b, c\) are particles and \(X\) stands for the unobserved remaining reaction products. We will present a general method which allows us to write the inclusive differential cross section \(d\sigma(a + b \to c + X)/d^3p_3\) as imaginary part of either a current-current or a field-field correlation function in a forward scattering amplitude \(a + b \to a + b\) in a modified theory. Let
\[
H = H_0 + H_I \tag{1.2}
\]
be the Hamiltonian of the original theory, with \(H_0\) and \(H_I\) the free and interaction parts, respectively. Then the modified theory is described by
\[
\tilde{H} = H_0 + \tilde{H}_I, \tag{1.3}\]
where \(\tilde{H}_I\), obtained from \(H_I\) in a well defined procedure, is discontinuous in time and nonlocal in space. The modified theory is constructed in such a way that both its incoming and outgoing states are equal to the incoming states of the original theory. Thus the S-matrix of the modified theory equals the unit operator. Here and in the following we always work in the Heisenberg picture of quantum mechanics.

Our article is organised as follows. In Sect. 2 we recall the basic relations for single inclusive cross sections.

In Sect. 3 we present our general formalism for the modified effective theory in the case of scalar fields. Quantum electrodynamics with massive photons and the theory of quarks interacting with abelian gluons are considered in Sect. 4. We discuss some properties of the modified Hamilton operator and derive a path integral representation for inclusive cross sections in the abelian gluon model. These techniques are then applied to the cross section \(e^+ + e^- \rightarrow q + X\) as a specific example. We compare our techniques with the Schwinger-Keldysh formalism [1], described e.g. in [2], and with Mueller’s treatment [3] of inclusive cross sections using the generalised optical theorem for \(3 \rightarrow 3\) scattering in Sect. 5 which contains also our conclusions.

2 Single inclusive cross sections

In this section we recall some basic relations for inclusive cross sections. Our notation follows [4, 5]. Let us consider a single-particle inclusive reaction, i.e.
\[
a(p_1) + b(p_2) \rightarrow c(p_3) + X(p_X). \tag{2.1}\]
To take a simple case, let \(a, b, c\) be spinless particles with masses \(m_a, m_b, m_c\). The c.m. energy squared is \(s = (p_1 + p_2)^2\). We assume \(c(p_3)\) to differ in type and/or momentum from \(a(p_1)\) and \(b(p_2)\). We use the covariant normalisation for our state vectors
\[
\langle a(p'_1) | a(p_1) \rangle = (2\pi)^3 \cdot 2p_1^0 \cdot \delta^{(3)}(p'_1 - p_1) \tag{2.2}\]
and similarly for \(b, c\). The S-matrix element for the reaction (2.1) is given as
\[
S_{fi} = \langle c(p_3), X(p_X), out | a(p_1), b(p_2), in \rangle \]
Here we have applied the reduction formula for particle $c$ in the final state. Let $\phi_c(x)$ be a suitable interpolating field for $c$ and $Z_c$, the corresponding wave function renormalisation constant. The current $j_c(x)$ is defined as

$$j_c(x) = (\Box + m_c^2) \phi_c(x).$$  

(2.4)

The $T$-matrix element is obtained from the $S$-matrix element via

$$S_{fi} = \delta_{fi} + i(2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_X)T_{fi},$$

$$T_{fi} = Z_c^{-1/2} \langle X(p_X), out | j_c(0) | a(p_1), b(p_2), in \rangle.$$  

(2.5)

The single-particle inclusive cross section $f_{inc}(p_3)$ is defined by

$$f_{inc}(p_3) := p_3^0 \frac{d^3p}{d^3p_3} (a + b - c + X)$$

$$= \frac{1}{4(2\pi)^3 w(s, m_a^2, m_b^2)} \sum_x$$

$$\langle x | a(p_1), b(p_2), in \rangle \langle j_c(x) | a(p_1), b(p_2), in \rangle,$$  

(2.6)

$$w(x, y, z) = |x^2 + y^2 + z^2 - 2xy - 2xz - 2yz|^{1/2}.$$  

(2.7)

In the usual way the sum over all states $\{X, out\}$ in (2.6) can be carried out using completeness and translational invariance

$$f_{inc}(p_3) = \frac{1}{4(2\pi)^3 w(s, m_a^2, m_b^2)} Z_c^{-1} \int d^4x \ e^{ip_3x}$$

$$\times \langle x | a(p_1), b(p_2), in \rangle \langle j_c(x) | a(p_1), b(p_2), in \rangle,$$  

(2.8)

$$= \frac{1}{2(2\pi)^3 w(s, m_a^2, m_b^2)} \int C(p_1, p_2, p_3),$$  

(2.9)

$$C(p_1, p_2, p_3) = i \int d^4x \ e^{ip_3x} M(x),$$  

(2.10)

$$M(x) = Z_c^{-1} \langle x | a(p_1), b(p_2), in \rangle \langle j_c(x) | a(p_1), b(p_2), in \rangle \theta(-x^0).$$  

(2.11)

Here $\theta(z)$ is the usual step function. Alternatively we can write

$$M(x) = Z_c^{-1} \langle x | a(p_1), b(p_2), in \rangle \langle j_c(x) | a(p_1), b(p_2), in \rangle \theta(-x^0)$$

$$\phi_c(x + z) a(p_1), b(p_2), in \rangle \theta(-x^0).$$  

(2.12)

Here the limit $y \to 0^-$, $z \to 0^-$ is to be understood as follows: We first require $y^0 < 0$ and $z^0 < 0$ and perform the differentiations with respect to $y$ and $z$. Afterwards we take the limit $y \to 0$ and $z \to 0$.

The amplitude $C(p_1, p_2, p_3)$ will play a central role in the following and we will be able to write it as a current-current, respectively a field-field correlation function in a forward scattering amplitude, but in a certain modified effective theory.

### 3 Modified effective theory for scalar fields

Let us assume that the basic dynamical variables of the original theory are the operators for unrenormalised scalar fields $\phi_i(x)$ and their conjugate canonical momenta $\Pi_i(x)$ ($i = 1, ..., N$). For simplicity we assume that $\Pi_i(x) = 0$. We denote $\phi_i(x)$, $\Pi_i(x)$ collectively as $\Phi(x)$. Let $H$ be the Hamiltonian of the system which we split into a free part $H_0$ and an interaction part $H_I$ which may depend explicitly on the time $t$, but should not involve time derivatives of $\Pi_i(x)$

$$H(t, \Phi(x, t)) = H_0(\Phi(x, t)) + H_I(t, \Phi(x, t)).$$  

(3.1)

Besides the interacting fields and momenta $\Phi$ free fields and momenta $\Phi(0)$ are considered with the corresponding Hamiltonian $H_0$. Here the mass parameters in $H_0$ are taken to be the ones of the asymptotic particles.

We assume now as usual (cf. e.g. [6,7]) that there exist unitary operators $U(t)$ that realize the time-dependent canonical transformations relating $\Phi$ to $\Phi(0)$

$$\Phi(x, t) = U^{-1}(t)\Phi(0)(x, t) U(t).$$  

(3.2)

Taking as boundary condition

$$\Phi(x, 0) = \Phi(0)(x, 0),$$  

(3.3)

we get

$$\frac{\partial}{\partial t} U(t) = -iH_I(t, \Phi(0)(x, t)) U(t),$$

$$U(0) = 1,$$  

(3.4)

$$U(t) = \sum_{n=0}^{\infty} (-i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 ... \int_0^{t_n-1} dt_n H_I(t_1) ... H_I(t_n),$$

$$H_I(t_j) \equiv H_I(t_j, \Phi(0)(x, t_j)).$$  

(3.5)

We define furthermore

$$U(t_2, t_1) = U(t_2) U^{-1}(t_1).$$  

(3.6)

For $t_2 \geq t_1$ we have

$$U(t_2, t_1) = T \exp \left[ -i \int_{t_1}^{t_2} dt' H_I(t', \Phi(0)(x, t')) \right],$$  

(3.7)

where $T$ means time-ordering.

Let us recall the LSZ formalism [9,6,7]. Assuming for simplicity the particles $a$ and $b$ to carry the quantum numbers of some fundamental hermitian scalar fields $\phi_a, \phi_b$, we define operators

$$A(p_1, x^0) = iZ_a^{-1/2} \int d^3x e^{ip_1x} \phi_a(x).$$  

(3.8)

Of course this is only true in the regularised theory, i.e. for finite ultraviolet cutoff