An analytical study of a periodically driven laser with a saturable absorber

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Abstract. We consider the rate equations for a laser with an intracavity saturable absorber and subject to a periodically modulated pump. By deriving simplified equations for a map valid for strongly pulsating regimes, analytical conditions are determined that specify the properties of both frequency-locked and unlocked behaviors. As the strength of the modulation is increased, quasiperiodic and period-doubling bifurcations are predicted. However, only the transition from locking to non-locking through a quasiperiodic bifurcation is possible for realistic values of the parameters. Our results are consistent with previous numerical and experimental studies of modulated lasers with a saturable absorber.

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1 Introduction

Lasers with an intracavity saturable absorber (LSA) have long been used to generate high-intensity pulses (see [1–4] for background and historical references). The absorber acts to limit the light output and hence prevent depletion of the active media. However, a threshold is reached such that the absorber becomes transparent and allows a rapid depletion of the now very strong population inversion and the emission of a pulse. The effect is similar to “Q-switching” the laser-cavity losses and hence the LSA is said to exhibit a “passive Q-switch” (PQS) behavior. The PQS output is of practical interest for applications that require extremely short (< 1 ns) high-peak-power (> 10 kW) pulses of light. The short pulse widths are useful for high-precision optical ranging with applications in automated production. The high peak output intensities are needed for efficient nonlinear frequency generation or ionization of materials, with applications in micro-surgery and ionization spectroscopy. Combined theoretical and experimental studies of PQS first considered gas lasers [5,6] and then concentrated on microchip solid state lasers and semiconductor lasers. Microchip lasers are small, easy to manipulate and offer high performances for the pulse width and/or peak-power [7,8]. Self-pulsing semiconductor lasers exhibit a high repetition rate which ranges from hundreds of megahertz to a few gigahertz [9,10]. They are interesting for telecommunication and for optical data storage using compact disc (CD) or digital versatile disc (DVD) systems [11–17].

Frequency jitter (stochastic variations in the frequency) in oscillating systems such as the LSA can be reduced by frequency-locking the system to an external drive source with more stable periodic output [18–20]; in lasers, this is often referred to as “linewidth narrowing”. The goal of achieving linewidth narrowing has motivated a number of studies of the “modulated LSA” (LMSA) [12,18–22]. In these works, the frequency-locking characteristics and the unlocked behavior of the LMSA have been investigated both numerically and experimentally. The unlocked behavior includes quasiperiodic output and chaos. Practically, we would like good locking properties for stable, tunable laser outputs. The interest of periodically modulated LSA for applications, the experimental observation of complex dynamics, and the numerical simulation of these phenomena using rate-equations, motivate analytical studies.

The main objective of this paper is to derive simple conditions that describe the locking region of the LMSA as a function of the modulation amplitude and frequency. These conditions show a variety a locking possibilities, emphasize the role of certain parameters, and predict bifurcation points possibly leading to complex (chaotic) dynamics. The formulation of the LSA dimensionless equations depends on the type of laser but is documented in the literature. See, for example, references [5,6] for CO$_2$ lasers, reference [24] for self-pulsating diode lasers, and reference [30] for microchip solid state lasers. The LMSA two-level atoms rate equations for the intensity of the laser
field $I$ and the population inversions of the active and passive regions, $D_1$ and $D_2$, are given by
\[
\frac{dI}{dt} = [D_1 + D_2 - 1]I,
\]
\[
\frac{dD_1}{dt} = \gamma_1[A_1(1 + \delta \cos(\omega t)) - (1 + I)D_1]
\]
\[
\frac{dD_2}{dt} = \gamma_2[A_2 - (1 + aI)D_2].
\]

The parameters $\gamma_1$ and $\gamma_2$ are the decay rates of $D_1$ and $D_2$ normalized by the cavity decay rate, respectively. $A_2 < 0$ is defined as the absorber pump parameter and $a$ describes the relative saturability of the absorber with respect to the active media. The first term in the equation for $D_1$ models a periodically modulated pump parameter where $A_1$, $\delta$ and $\omega$ represent its averaged value, its modulation amplitude, and its normalized modulation frequency, respectively. Typical values of the laser parameters are documented in [5,6] for CO$_2$ lasers, in [23] for microchip solid state lasers, and in [24] for semiconductor lasers. The range of values of these parameters may depend on the type of laser but they all exhibit small values of $\gamma_1$ and $\gamma_2$ ($10^{-5}$ to $10^{-3}$). The strongly pulsating behavior of PQS is directly related to these small values of $\gamma_1$ and $\gamma_2$ [29]. For self-pulsing semiconductor lasers, (1) needs to be supplemented by additional terms modeling nonlinear gain saturation, nonlinear damping of $\gamma$, and cross-diffusion of the carriers between active and passive regions. However, none of these additional effects are responsible for the generation of PQS [24]. Self-pulsation appears through a bifurcation mechanism which we briefly review. If $\delta = 0$, the domain of pulsating intensities is bounded by either the laser first threshold and a Hopf bifurcation point or by two Hopf bifurcation points. In the latter case, the low intensity Hopf bifurcation point is located very close to the laser first threshold and the leading approximation of the domain of self-pulsation is mathematically the same for both cases in the limit of small values of $\gamma_1$ and $\gamma_2$. In terms of $A_1$, the domain of self-pulsation is approximately given by
\[
A_{\text{th}} < A_1 < A_{\text{H}}
\]
where $A_{\text{th}} \equiv 1 - A_2$ is defined as the laser first threshold and $A_{\text{H}}$ corresponds to a high intensity Hopf bifurcation point; the expression for $A_{\text{H}}$ is not needed for our analysis. The laser intensity oscillations are strongly pulsating near $A_{\text{th}}$ and exhibits long interpulse periods. They progressively become smoother as we increase $A_1$ and approach $A_{\text{H}}$.

Equation (1) simplifies if we adiabatically ($\gamma_2 \gg \gamma_1$) eliminate $D_2$. From equation (1), we then obtain the following two equations
\[
\frac{dI}{dt} = \left[D + \frac{A_2}{1 + aI} - 1\right]I,
\]
\[
\frac{dD_1}{dt} = \gamma[A(1 + \delta \cos(\omega t)) - (1 + I)D]
\]
where $D = D_1$, $\gamma = \gamma_1$ and $A = A_1$. Equation (3) is studied in detail in [22] and has been shown to possess many dynamical features of equation (1). An adiabatic elimination of $D_2$ is also proposed in [19] from the four-level atoms rate equations. We have studied the PQS regimes of both equation (1) and equation (3) and did not find qualitative differences for the PQS locking conditions. In this paper, we concentrate on the pulsating solutions of equation (3) appearing for $\gamma$ small. In this limit, the domain of pulsating intensities is given by (2) where $A_{\text{th}} = 1 - A_2$ and $A_{\text{H}} \approx [-A_2/(a\gamma)]^{1/2}$.

Lauterborn and Eick [22] have shown that the LSA equations can be written as a perturbed Hamiltonian. This suggests the possibility of using methods such as averaging or subharmonic-Melnikov theory [26,27] to analyze the LMSA. However, the dissipation is not uniformly small over the complete period of the orbit, and this prevents the use of the averaging type methods.

Our analysis will make explicit use of the pulsating nature of PQS where the high-intensity pulses are followed by a long time during which the intensity is close to zero. By using the method of matched asymptotic expansions [28], we obtain asymptotic approximations to the dynamics in each regime. This enables us to construct a map for the amplitude and period from one pulse to the next. Fixed points of the map then correspond to frequency-locked solutions of (3).

The derivation of the map is given in Section 2 and the analysis of its fixed points is described in Section 3. In Section 4, we discuss the physical implications of our results.

### 2 A map describing PQS with modulation

To analyze the LSA in the regime of PQS, we take advantage of the fact that the high-intensity pulses are followed by a period during which the intensity is nearly zero. We will use different asymptotic approximations for (3) to analyze each regime in the spirit of “boundary-layer” analysis [28]. This approach has been used to analyze the period and maximum amplitude of the free PQS in [29,30]. Our problem is however more complicated because we consider a time-periodic pump. For the simpler problem of a modulated class-B laser, this has been done in [31,32]. The results of the analysis are given by equations (8, 9) for the “MPQS-map”. This map determines the time $T_{n+1}$ and inversion $D_{n+1}$ of the next pulse given $T_n$ and $D_n$ of the present pulse. Increasing $n$ allows us to determine how the period of the pulses and the inversion (from which we can determine the intensity) evolve in time. Since the mathematical analysis is similar to the one described in [29], we only emphasize the analytical differences.

#### 2.1 The interpulse regime

We first investigate the interpulse regime during which $I \ll 1$ and $D$ increases from $D(T_0)$ to $D(T_1)$ (Fig. 1). In (3), we let $T \equiv \gamma t$ and assume $I \ll 1$. The equation for