A posteriori error analysis of space-time finite element discretizations of the time-dependent Stokes equations

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Abstract We present a novel a posteriori error analysis of space-time finite element discretizations of the time-dependent Stokes equations. Our analysis is based on the equivalence of error and residual and a suitable decomposition of the residual into spatial and temporal contributions. In contrast to existing results we directly bound the error of the full space-time discretization and do not resort to auxiliary semi-discretizations. We thus obtain sharper bounds. Moreover the present analysis covers a wider range of discretizations both with respect to time and to space.

Keywords A posteriori error analysis · Error indicators · Time-dependent Stokes equations · Space-time finite elements · $\theta$-scheme

Mathematics Subject Classification (2000) 65N30 · 65N15 · 65J10

1 Introduction

We consider the time-dependent Stokes equations

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\begin{align*}
\partial_t u - \nu \Delta u + \nabla p &= f & \text{in } \Omega \times (0, T), \\
\text{div } u &= 0 & \text{in } \Omega \times (0, T), \\
u &= 0 & \text{on } \Gamma \times (0, T), \\
\mathbf{u}(\cdot, 0) &= \mathbf{u}_0 & \text{in } \Omega
\end{align*}
$$

(1.1)

in a bounded space-time cylinder with a polygonal cross-section $\Omega \subset \mathbb{R}^d$, $d \geq 2$, having a Lipschitz boundary $\Gamma$. The final time $T$ is arbitrary, but kept fixed in what
follows. The unknowns are the velocity $u$ and the pressure $p$; the data are the distribution $f$ which represents a density of body forces and the initial velocity $u_0$, while the viscosity $\nu$ is a positive constant.

Problem (1.1) could be formulated as a heat equation in the space $V = \{u \in H^1_0(\Omega)^d : \operatorname{div} u = 0\}$ of solenoidal velocity fields. Since this space has the same analytical properties as $H^1_0(\Omega)^d$, the results of [22] would immediately yield a posteriori error estimates for space-time discretizations of problem (1.1). This approach, however, is not feasible since virtually all finite element discretizations used in practice are non-conforming in the sense that the discrete velocities are either discontinuous and thus not contained in $H^1_0(\Omega)^d$ or are not solenoidal and thus not contained in $V$. Therefore, we must develop particular techniques for the a posteriori error analysis of problem (1.1).

A first step in this direction is the a posteriori error analysis of [5]. There, the non-conformity of the discretization is taken into account by separately estimating the errors of suitable semi-discretizations with respect to time and to space. As a consequence the error indicator of [5] gives bounds for the sum of the errors of the semi-discretizations which usually is much larger than the error of the full space-time discretization.

In this note we overcome this drawback. The crucial point here is to establish the equivalence of error and residual (cf. Proposition 4.2 below). This is not a trivial task due to the non-conformity of the discretization. It is achieved with the help of a suitable Stokes projection. Once the equivalence of error and residual is established, the remaining analysis follows the lines of [22].

The error estimates of Theorem 8.1 below are weaker than those of [5, Theorem 5.1] in that they do not provide separate bounds for the error $p - p_I$ of the pressure approximation. They are stronger in that they apply to a larger class of discretizations, require weaker assumptions on the spatial and temporal partitions and yield direct control on the error of full space-time discretizations.

This note is organized as follows: In Sects. 2 and 3, we present the variational formulation and space-time finite element discretization of problem (1.1). In Sect. 4, we then prove the crucial equivalence of error and residual. Next, in Sect. 5, we show that the residual can be decomposed into spatial and temporal contributions and that the norm of the sum of these contributions is equivalent to the sum of the individual norms. Thus, these contributions can be estimated separately which is the object of Sects. 6 and 7. Finally, in Sect. 8, we collect all results and obtain the final a posteriori error estimates for problem (1.1).

2 Variational formulation

For every open bounded domain $\omega \subset \mathbb{R}^d$ with Lipschitz boundary $\gamma$ and every integer $k \geq 1$, we denote by $L^2(\omega)$, $H^k(\omega)$ and $L^2(\gamma)$ the standard Lebesgue and Sobolev spaces equipped with the usual norms $\|\cdot\|_{L^2}$, $\|\cdot\|_{H^k(\omega)}$ and $\|\cdot\|_{\gamma}$, cf. [1]. If $\omega = \Omega$, we will drop the subscript $\Omega$. As usual, $H^1_0(\Omega)$ denotes the subset of all functions in $H^1(\Omega)$ which vanish on the boundary $\Gamma$. Its dual space is denoted by $H^{-1}(\Omega)$. 