Numerical computation of the optimal feedback law for nonlinear infinite time horizon control problems

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Abstract. An approach to the numerical computation of the optimal feedback law for nonlinear infinite time horizon control problems is presented and tested. For a sequence of control systems with truncated time horizon the canonical equations are solved by an adapted multiple shooting algorithm. Interpolation of computed costates gives a feedback law on a prescribed region of the state space. Numerical findings indicate applicability of this procedure to problems in higher dimensions.

1 Introduction and preliminaries

This paper deals with the numerical treatment of certain nonlinear infinite time horizon optimal control problems. The investigation is restricted to control problems possessing a unique optimal control which can be easily expressed in terms of state and costate. Our approach is based on computation of costates corresponding to states of the given control system in a prescribed region of $\mathbb{R}^n$. Approximations for the optimal feedback law are obtained by interpolation of these costates with respect to the states. The assignment from a given state to the corresponding costate is based on interpreting each as one proper half of the initial state of the canonical system associated with the given control problem. Since the given control problem has infinite time horizon, the actual implementation of this assignment is done by solving appropriate boundary value problems for a sequence of canonical systems associated with control problems that are governed by the same state equations as the given one, but with truncated time horizon and an auxiliary valuation of the final state. The purpose of the investigation summarized in this paper is to check whether the approach described
actually works and specifically whether it works for higher-dimensional systems. No convergence theory is developed in this paper and error analysis is addressed only in a short concluding remark.

Notation and some well-known facts (see e.g., [1]) are collected below. Sect. 2 contains in its first paragraph a detailed description of a shooting algorithm adapted for solving boundary value problems for the canonical system on a sequence of increasing time horizons. In the second paragraph feedback laws are constructed by interpolation. In Sect. 3 applicability of the approach is demonstrated by means of five examples in various dimensions. Finally, in Sect. 4, some disadvantages are considered and remarks concerning an error analysis and possible improvements are made.

Let \( n, q \in \mathbb{N} \), \( q \leq n \), let \( D \subset \mathbb{R}^n \) be a simply connected domain, \( x_0, \bar{x} \in D \), \( h \in C^1(D; \mathbb{R}) \) with \( h(\bar{x}) = 0 \) and \( h(x) > 0 \) for all \( x \neq \bar{x} \), \( f \in C^1(D; \mathbb{R}^n) \) with \( f(\bar{x}) = 0 \), \( g \in C^1(D; \mathbb{R}^n) \) not identically zero and let \( u : [0, \infty) \rightarrow \mathbb{R}^q \) be a control function. Consider the control problem:

\[
\begin{align*}
\text{Minimize} & \quad J(x_0, u) = \int_0^\infty \left( h(x(t)) + u(t)^\top u(t) \right) dt \\
\text{subject to} & \quad \dot{x}(t) = f(x(t)) + g(x(t))u(t), \quad t \geq 0, \\
& \quad x(0) = x_0.
\end{align*}
\]

If a minimizing control function exists, it will be called an optimal control and denoted by \( u^*(t) \). The solution of \( \dot{x}(t) = f(x(t)) + g(x(t))u^*(t) \), \( x(0) = x_0, t \geq 0 \), will be called an optimal trajectory and denoted by \( x^*(t) \).

For \( p \in \mathbb{R}^n, x \in D, u \in \mathbb{R}^q \) the Hamiltonian for the control problem is defined by

\[
H(x, p, u) = p^\top (f(x) + g(x)u) + h(x) + u^\top u.
\]

Suppose that the optimal control problem has a solution and that \( u^* \) is an optimal control. Then, by Pontryagin’s maximum principle, \( \partial H / \partial u_i (x, p, u^*) = 0 \) for \( i = 1, \ldots, q \). This system of equations has the unique solution

\[
u^* = -\frac{1}{2} g(x)^\top p,
\]

and the Hamiltonian at \( u^* \) is given by

\[
H^*(x, p) = H(x, p, u^*) = p^\top f(x) - \frac{1}{4} p^\top g(x)g(x)^\top p + h(x).
\]